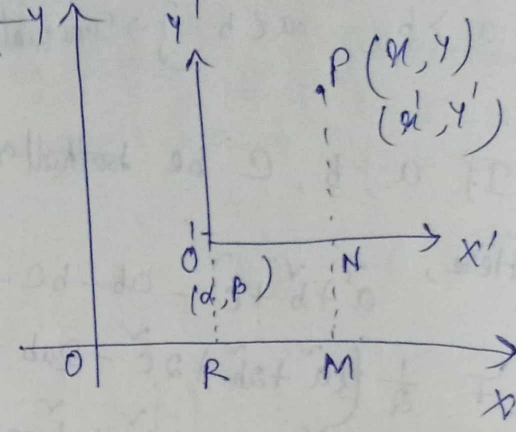
Geometry :- Transformation of co-ordinates

1) Translation (Shifting of origin) :-



$OM = x \qquad PM = y$

$O'N = x' \qquad PN = y'$

$OR = \alpha \qquad O'R = \beta$

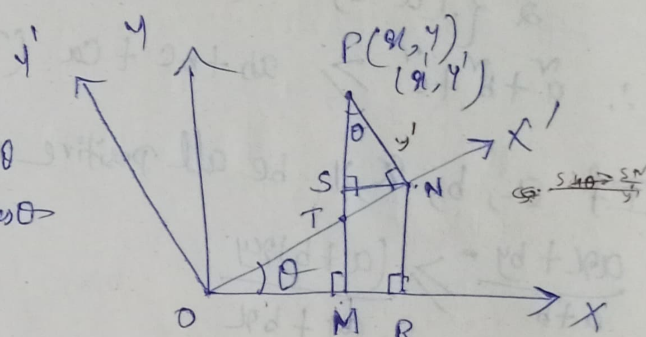
$x = OM = OR + RM = \alpha + x'$

or, $\boxed{x = x' + \alpha}$

$y = PM = PN + NM = y' + O'R = y' + \beta$

$\boxed{y = y' + \beta}$

2) Rotation of axis :-



$OM = x, \; PM = y, \; SN = y' \sin\theta$

$ON = x', \; PN = y', \; PS = y' \cos\theta$

ΔPNS,

$\sin\theta = \frac{SN}{PN} = \frac{SN}{y'} \Rightarrow SN = y' \sin\theta$

$\cos\theta = \frac{PS}{PN} = \frac{PS}{y'}$

or, $PS = y' \cos\theta$

from ΔONR,

$\cos\theta = \frac{OR}{ON} = \frac{OR}{x'}$

$OR = x' \cos\theta$

$\sin\theta = \frac{NR}{ON} = \frac{NR}{x'}$

$NR = x' \sin\theta$

$\therefore x = OM = OR - MR = x' \cos\theta - SN = x' \cos\theta - y' \sin\theta$

$\therefore y = PM = PS + SM = y' \cos\theta + x' \sin\theta$

$\boxed{x = x' \cos\theta - y' \sin\theta}$ —— (i)

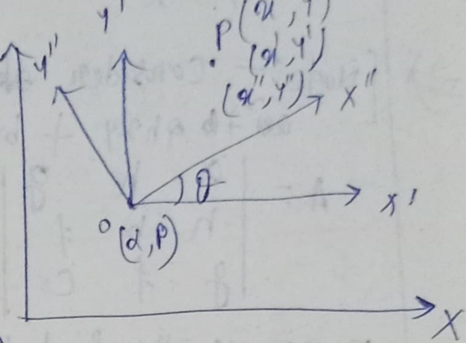
$\boxed{y = y' \cos\theta + x' \sin\theta}$ —— (ii) from (i) and (ii),

$x' = x\cos\theta + y\sin\theta$

$y' = -x\sin\theta + y\cos\theta$

	x'	y'
x	$\cos\theta$	$-\sin\theta$
y	$\sin\theta$	$\cos\theta$

3) Translation followed by rotation :-



$$\boxed{\begin{aligned} x &= \alpha + x''\cos\theta - y''\sin\theta \\ y &= \beta + x''\sin\theta + y''\cos\theta \end{aligned}}$$

$x' = x''\cos\theta - y''\sin\theta$

$y' = x''\sin\theta + y''\cos\theta$

$x - \alpha = x''\cos\theta - y''\sin\theta$

$y - \beta = x''\sin\theta + y''\cos\theta$

$x = \alpha + x''\cos\theta - y''\sin\theta$

$y = \beta + x''\sin\theta + y''\cos\theta$

1) Find the translation which transforms the eqn $x^2 + y^2 - 2x + 14y + 20 = 0$ into $x^2 + y^2 - 30 = 0$

⇒ The given eqn, $x^2 + y^2 - 2x + 14y + 20 = 0$ ——(i)

Let, Origin be shift to the point (α, β)

∴ By law of translation we have, $[(x', y')$ being current co-ordinate$]$

$x = x' + \alpha$ and $y = y' + \beta$,

∴ from (i), $(x' + \alpha)^2 + (y' + \beta)^2 - 2(x' + \alpha) + 14(y' + \beta) + 20 = 0$

or, $x'^2 + y'^2 + (2\alpha - 2)x' + (2\beta + 14)y' + (\alpha^2 + \beta^2 - 2\alpha + 14\beta + 20) = 0$ ——(ii)

Equating the coefficients of x' and y' to zero we have,

$2\alpha - 2 = 0$ and $2\beta + 14 = 0$

$\Rightarrow \alpha = 1$ $\quad \Rightarrow \beta = -7$

Putting the values of α and β into (ii)

$$x'^2 + y'^2 + (1+49-2-14\cdot 7+20) = 0$$

$$\Rightarrow x'^2 + y'^2 - 30 = 0$$

2) [Note:— Consider the general 2nd degree eq^n

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + C = 0 \quad \text{—(i)}$$

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & C \end{vmatrix}, \quad D = \begin{vmatrix} a & h \\ h & b \end{vmatrix}$$

To remove the first degree terms, we are to shift the origin to the point (α, β) where α, β is given by $a\alpha + h\beta + g = 0$ and $h\alpha + b\beta + f = 0$.
and the transfered eq^n is $ax'^2 + 2hx'y' + by'^2 + \frac{\Delta}{D} = 0$]

2) Remove the first degree terms from the eq^n $12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0$ and find the transfered eq^n.

$\Rightarrow$ The given eq^n,

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0 \quad \text{—(i)}$$

Here, $a = 12$, $b = 2$, $g = \frac{11}{2}$, $f = -\frac{5}{2}$, $C = 2$, $h = -5$

$$\therefore \Delta = \begin{vmatrix} 12 & -5 & \frac{11}{2} \\ -5 & 2 & -\frac{5}{2} \\ \frac{11}{2} & -\frac{5}{2} & 2 \end{vmatrix}$$

$$= 12\left(4 - \frac{25}{4}\right) + 5\left(-10 + \frac{55}{4}\right) + \frac{11}{2}\left(\frac{25}{2} - 11\right)$$

$$= -12 \times \frac{9}{4} + 5 \times \frac{15}{4} + \frac{11}{2} \times \frac{3}{2}$$

$$= \frac{\cancel{492 + 75 + 33}}{4} = \cancel{\frac{600}{4 \times}} \cancel{= 150}$$

$$= 0$$

$$\therefore D = \begin{vmatrix} 12 & -5 \\ -5 & 2 \end{vmatrix} = 24 - 25 = -1$$

Let, Origin be shift to the point (α, β)

$\therefore 12\alpha - 5\beta + \frac{11}{2} = 0$ — (ii)

$-5\alpha + 2\beta - \frac{5}{2} = 0$ — (iii)

$$\frac{\alpha}{\frac{25}{2} - 11} = \frac{\beta}{-\frac{55}{2} + 30} = \frac{1}{24 - 25}$$

$$\Rightarrow \alpha = \frac{25-22}{-2}, \quad \beta = \frac{30 - \frac{55}{2}}{-1} = \frac{60-55}{-2}$$

$$= -\frac{3}{2} \qquad\qquad = -\frac{5}{2}$$

$\therefore$ The transfered eq^n, $12x'^2 - 10x'y' + 2y'^2 \cancel{+c} = 0$

3) Remove the xy term from the eq^n $9x^2 - 2\sqrt{3}xy + 7y^2 = 0$

$\Rightarrow$ The given eq^n, $9x^2 - 2\sqrt{3}xy + 7y^2 = 0$ — (i)

To remove the xy term let us rotate the axes through an angle θ.

$\therefore$ We have the law of rotation,

$x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$

$\therefore$ from (i),

$$9(x'\cos\theta - y'\sin\theta)^2 - 2\sqrt{3}(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + 7(x'\sin\theta + y'\cos\theta)^2 = 0$$

$$\Rightarrow (9\cos^2\theta - 2\sqrt{3}\sin\theta\cos\theta + 7\sin^2\theta)x'^2 + (-18\sin\theta\cos\theta - 2\sqrt{3}\cos^2\theta + 2\sqrt{3}\sin^2\theta + 14\sin\theta\cos\theta)x'y'$$

$$+ (9\sin^2\theta + 2\sqrt{3}\sin\theta\cos\theta + 7\cos^2\theta)y'^2 = 0 \quad \text{(ii)}$$

equating the coefficient of $x'y'$ to zero we have,

$$-18\sin\theta\cos\theta - 2\sqrt{3}\cos^2\theta + 2\sqrt{3}\sin^2\theta + 14\sin\theta\cos\theta = 0$$

or, $\tan 2\theta = -\sqrt{3}$

or, $2\theta = 120$

or, $\theta = 60 = \frac{\pi}{3}$

Putting the value of θ into (ii)

$$\left(9\cdot\frac{1}{4} - 2\sqrt{3}\,\frac{1}{2}\cdot\frac{\sqrt{3}}{2} + 7\cdot\frac{3}{4}\right)x'^2 + \left(9\times\frac{3}{4} + 2\sqrt{3}\times\frac{1}{2}\times\frac{\sqrt{3}}{2} + 7\times\frac{1}{4}\right)y'^2 = 0$$

or, $\left(\frac{9}{4} - \frac{6}{4} + \frac{21}{4}\right)x'^2 + \left(\frac{27}{4} + \frac{6}{4} + \frac{7}{4}\right)y'^2 = 0$

or, $\frac{24}{4}x'^2 + \frac{40}{4}y'^2 = 0$

or, $6x'^2 + 10y'^2 = 0$

or, $3x'^2 + 5y'^2 = 0$

[Note:– To remove the xy term from the eq^n $ax^2+2hxy+by^2=0$, axes are turned through an angle θ, where θ is given by $\tan 2\theta = \frac{2h}{a-b}$]

(11) Law of invariants :–

Due to rotation of axes let the eq^n $ax^2+2hxy+by^2=0$ be transfered to $a'x'^2+2h'x'y'+b'y'^2=0$, then

i) $a+b = a'+b'$ and

ii) $ab-h^2 = a'b'-h'^2$

Let, the axes be turned through an angle θ.

We have the law of rotation,

$\therefore$ $x = x'\cos\theta - y'\sin\theta$

and $y = x'\sin\theta + y'\cos\theta$

$\therefore$ $ax^2+2hxy+by^2=0$ gives,

$$a(x'\cos\theta - y'\sin\theta)^2 + 2h(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + b(x'\sin\theta + y'\cos\theta)^2 = 0 \quad \text{—(i)}$$

$\therefore$ By the given condition

$a' = a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta$ —(ii)

$b' = a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta$

$$2h' = -2a\sin\theta\cos\theta + 2h\cos^2\theta - 2h\sin^2\theta + 2b\sin\theta\cos\theta \quad \text{--- (iii)}.$$

$\therefore$ from (i) and (ii),

$$a' + b' = a + 0 + b = a + b$$

$\therefore$ from (i),

$$2a' = a(1+\cos 2\theta) + 2h\sin 2\theta + b(1-\cos 2\theta)$$
$$= (a+b) + (a-b)\cos 2\theta + 2h\sin 2\theta \quad \text{--- (iv)}$$

from (ii),

$$2b' = a(1-\cos 2\theta) - 2h\sin 2\theta + b(1+\cos 2\theta)$$
$$= (a+b) - (a-b)\cos 2\theta - 2h\sin 2\theta \quad \text{--- (v)}$$

and from (iii),

$$2h' = (b-a)\sin 2\theta + 2h\cos 2\theta \quad \cancel{\text{(vi)}}$$
$$= 2h\cos 2\theta - (a-b)\sin 2\theta \quad \text{--- (vi)}$$

from (iv) and (v),

$$4a'b' = (a+b)^2 - \{(a-b)\cos 2\theta + 2h\sin 2\theta\}^2 \quad \text{--- (vii)}$$

squaring both sides of $(vi)^2$ we have,

$$4h'^2 = \{2h\cos 2\theta - (a-b)\sin 2\theta\}^2 \quad \text{--- (viii)}$$

subtracting (viii) from (vii),

$$4a'b' - 4h'^2 = (a+b)^2 - \{(a-b)\cos 2\theta + 2h\sin 2\theta\}^2 - \{2h\cos 2\theta - (a-b)\sin 2\theta\}^2$$

$$= (a+b)^2 - \{(a-b)^2 + 4h^2\}$$
$$= (a+b)^2 - (a-b)^2 - 4h^2$$
$$= 4ab - 4h^2$$

or, $a'b' - h'^2 = ab - h^2$.

[Note :- If the axes be turned through an angle $\theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b}$ then the eq^n $ax^2 + 2hxy + by^2 + c = 0$ is transfered to

$a'x'^2 + b'y'^2 + c = 0$

Where, $a' + b' = a + b$

and $a'b' = ab - h^2$

1) Remove the xy term from the eq^n $x^2 + 2\sqrt{3}xy - y^2 = 4$

⇒ The given eq^n,

$$x^2 + 2\sqrt{3}xy - y^2 - 4 = 0 \quad \text{——— (i)}$$

To remove the xy term we are to rotate the axes through an angle $\theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b}$

$$= \frac{1}{2}\tan^{-1}\frac{2\sqrt{3}}{2}$$

$$= \frac{\pi}{6}$$

due to rotation through this angle, let (i) transfered to

$$ax'^2 + by'^2 - 4 = 0 \quad \text{——— (ii)}$$

∴ Law of invarients we have,

$$a + b = 0 \quad \text{——— (iii)}$$

$$ab = -1 - 3 = -4 \quad \text{——— (iv)}$$

from (iii), $a = -b$

∴ from (iv), $-a^2 = -4$

$a = \pm 2$

when $a = 2$ and when $a = -2$

$b = -2$ $\qquad$ $b = 2$

∴ The transfered eq^n, $x'^2 - y'^2 = 2$

General 2nd degree eq^n

Consider the general 2nd degree eq^n $ax^2+2hxy+by^2+2gx+2fy+c=0$ —(i)

Let, $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ and $D = \begin{vmatrix} a & h \\ h & b \end{vmatrix} = ab-h^2$

$= abc+2fgh-af^2-bg^2-ch^2$

1) i) If $\Delta = 0$ and $D \neq 0$ then (i) represents pair of intersecting straight lines.

ii) If $\Delta = 0$ and $D = 0$ then (i) represents pair of parallal straight lines.

2) for $\Delta \neq 0$

i) If $\Delta \neq 0$ and $D > 0$ then (i) represents an ellipes.

ii) If $\Delta \neq 0$ and $D < 0$ then (i) represents hyperbola.

iii) If $\Delta \neq 0$ and $D = 0$ then (i) represents parabola.

3) If $a = b$ and $h = 0$ then (i) represents a circle.

[Note :- If (i) represents a central conic and (α, β) be it's centre, Then α, β is given by $a\alpha+h\beta+g=0$ and $h\alpha+b\beta+f=0$

Canonical form :-

1) Transform the following eq^n to it's canonical form —
$3x^2-2xy+3y^2-4x-4y-12=0$

⇒ The given eq^n, $3x^2-2xy+3y^2-4x-4y-12=0$ —— (i)

Here, $a=3, b=3, c=-12, f=-2, g=-2, h=-1$

$\therefore \Delta = \begin{vmatrix} 3 & -1 & -2 \\ -1 & 3 & -2 \\ -2 & -2 & -12 \end{vmatrix} = 3(-36-4)+1(12-4)-2(2+6)$
$= -128$

and

$D = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 9-1 = 8$

Since, $\Delta \neq 0$ and $D > 0$, (i) represents an ellipes.

Let, (α, β) be the centre of the ellipse

$\therefore 3\alpha - \beta - 2 = 0$ —(ii)

$-\alpha + 3\beta - 2 = 0$ —(iii)

Solving (ii) and (iii) we have,

$\alpha = 1, \ \beta = 1$

$\therefore$ The centre of the ellipse is $(1,1)$

Shifting the origin to the point $(1,1)$, eq^n (i) transffered to

$$3x'^2 - 2x'y' + 3y'^2 + \frac{\Delta}{D} = 0$$

$$\text{or, } 3x'^2 - 2x'y' + 3y'^2 - 16 = 0 \quad \text{— (iv)}$$

To remove the $x'y'$ term let us rotate the axes through an angle θ where θ is given by $\tan 2\theta = \frac{2h}{a-b} = \frac{-2}{3-3}$

$$\therefore \theta = \frac{\pi}{4}$$

Let, (iv) becomes $aX^2 + bY^2 - 16 = 0$ — (v)

$\therefore$ By the law of invariants we have,

$$\left.\begin{aligned} a+b &= 3+3 = 6 \\ ab &= 3\cdot 3 - (-1)^2 = 8 \end{aligned}\right\} \text{— (vi)}$$

$\therefore (a-b)^2 = (a+b)^2 - 4ab = 36 - 32 = 4$

or, $a - b = \pm 2$

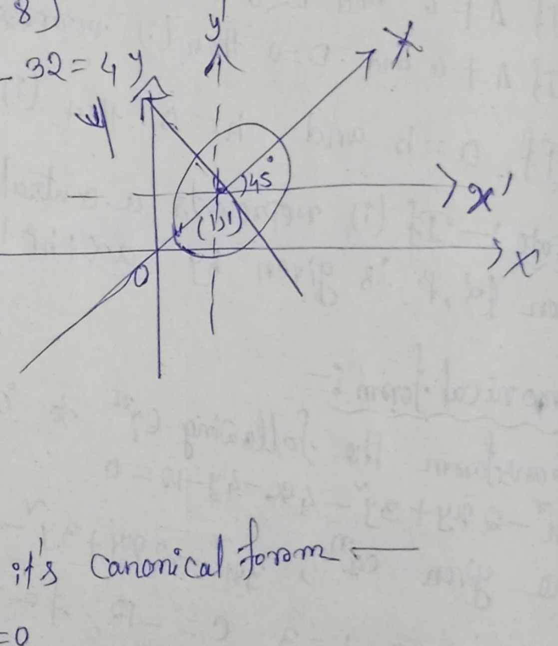
$\therefore a = 4, \ b = 2$

or, $a = 2, \ b = 4$

$\therefore$ from (iii), $4X^2 + 2Y^2 - 16 = 0$

$$\Rightarrow \frac{X^2}{4} + \frac{Y^2}{8} = 1$$

$$\text{or, } \frac{X^2}{8} + \frac{Y^2}{4} = 1$$



2) Reduce the following eq^n to it's canonical form:

$$x^2 - 6xy + y^2 - 4x - 4y + 12 = 0$$

$\Rightarrow$ The given eq^n is $x^2 - 6xy + y^2 - 4x - 4y + 12 = 0$ — (i)

Here, $a = 1, b = 1, c = 12, h = -3, f = -2, g = -2$

$\therefore \Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 1\cdot(1)(12) + 2\cdot(-2)(-2)(-3) - 1\times 4 - 1\times 4 - 12\times 9$

$= 12 - 24 - 4 - 4 - 108 = -128$

$D = -8$

Since, $\Delta \neq 0$ and $D < 0$, (i) represents a hyperbola.

Let, (α, β) be the centre of the hyperbola.

$\therefore \alpha - 3\beta - 2 = 0$ —(ii)

$-\alpha + \beta - 2 = 0$ —(iii)

Solving (ii) and (iii) we have,

$\alpha = -1, \beta = -1$

Shifting origin to the point $(-1,-1)$, (i) becomes,

$$x'^2 - 6x'y' + y'^2 + \frac{\Delta}{D} = 0$$

or, $x'^2 - 6x'y' + y'^2 + 16 = 0$ —(iv)

To remove the $x'y'$ term, let us rotate the axes through an angle θ, where θ is given by, $\tan 2\theta = \frac{2h}{a-b} = \frac{-6}{1-1}$

or, $\theta = \frac{\pi}{4}$

$\therefore$ (iv) becomes $aX^2 + bY^2 + 16 = 0$ —(v)

$\therefore$ By the law of invarients,

$$\left.\begin{aligned} a+b &= 1+1 = 2 \\ ab &= 1\cdot1 - (-3)^2 = 1 - 9 = -8 \end{aligned}\right\} \text{—(vi)}$$

$\therefore (a-b)^2 = (a+b)^2 - 4ab = 4 + 32 = 36$

or, $a - b = \pm 6$

$\therefore a = 4, b = -2$

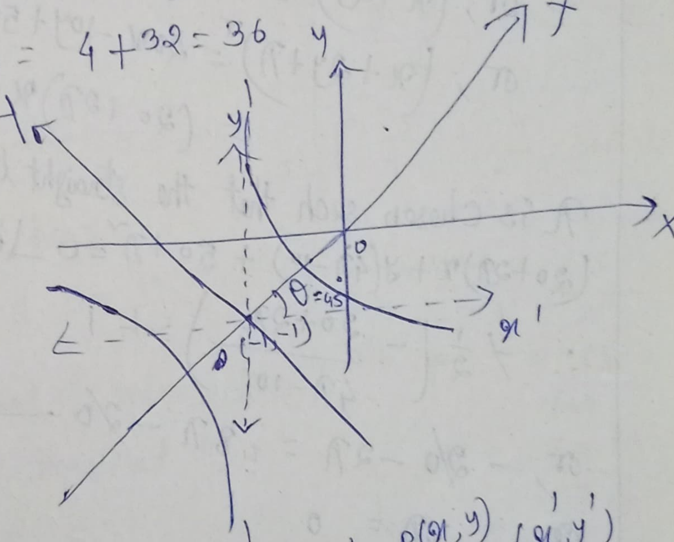
or, $a = -2, b = 4$

$\therefore$ from (v),

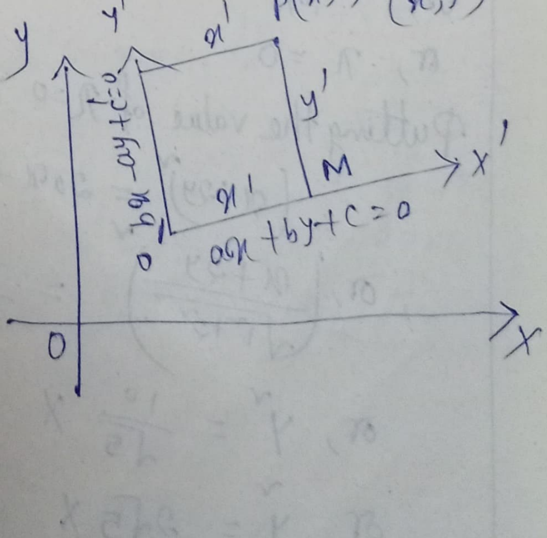
$$4X^2 - 2Y^2 + 16 = 0$$

$\Rightarrow \frac{X^2}{-4} + \frac{Y^2}{8} = 1$

or, $\frac{X^2}{8} - \frac{Y^2}{4} = 1$



General Orthogonal transformation :-



$\therefore O'M = x', PM = y'$

$\therefore x' = \frac{bx - ay + c'}{\sqrt{a^2 + b^2}}$

and $y' = \frac{ax + by + c}{\sqrt{a^2 + b^2}}$

3) Reduce the eqⁿ $x^2+4xy+4y^2-20x+10y-50=0$ to it's Canonical form.

⇒ The given eqⁿ, $x^2+4xy+4y^2-20x+10y-50=0$ —(i)

Here, $a=1$, $b=4$, $c=-50$, $f=5$, $g=-10$, $h=2$

$\therefore \Delta = abc+2fgh-af^2-bg^2-ch^2$

$= -200-200-25-400+200 = -625$

$D = ab-h^2 = 4-4=0$

Since, $\Delta \neq 0$ and $D=0$, (i) represents a parabola.

Eqⁿ (i) can be written as,

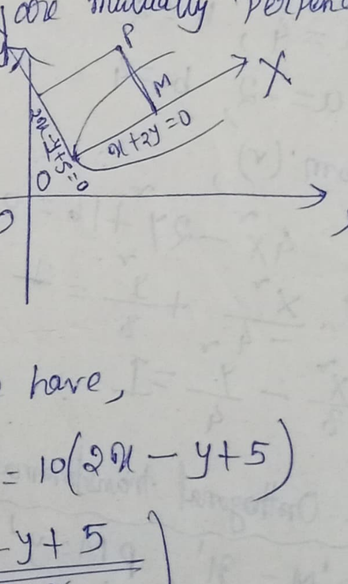
$$x^2+4xy+4y^2 = 20x-10y+50$$

or, $(x+2y)^2 = 20x-10y+50$

or, $(x+2y+\lambda)^2 = 20x-10y+50+2\lambda(x+2y)+\lambda^2$

$= (20+2\lambda)x+y(4\lambda-10)+50+\lambda^2$ —(ii)

λ is chosen such that the straight lines $x+2y+\lambda=0$ —(iii) and $(20+2\lambda)x+y(4\lambda-10)+50+\lambda^2=0$ —(iv) are mutually perpendicular



$\therefore -\frac{1}{2}\left(-\frac{20+2\lambda}{4\lambda-10}\right) = -1$

or, $-20-2\lambda = 8\lambda-20$

or, $10\lambda = 0$

or, $\lambda = 0$

Putting the value of $\lambda=0$ into (ii) we have,

$$(x+2y)^2 = 20x-10y+50 = 10(2x-y+5)$$

or, $\left(\frac{x+2y}{\sqrt{1^2+2^2}}\right)^2 = \frac{10}{\sqrt5}\left(\frac{2x-y+5}{\sqrt{2^2+1^2}}\right)$

or, $Y^2 = \frac{10}{\sqrt5}X$, where $Y = \frac{x+2y}{\sqrt{1^2+2^2}}$

$X = \frac{2x-y+5}{\sqrt{2^2+1^2}}$

or, $Y^2 = 2\sqrt5\, X$

4) Reduce the following eq^n $6x^2-5xy-6y^2+14x+5y+4=0$ to it's Canonical form.

$\Rightarrow$ The given eq^n,

$$6x^2-5xy-6y^2+14x+5y+4=0 \quad \text{---(i)}$$

Here, $a=6$, $b=-6$, $c=4$, $h=-\frac{5}{2}$, $g=7$, $f=\frac{5}{2}$

$\therefore \Delta = abc+2fgh-af^2-bg^2-ch^2$

$= -144 - \frac{175}{2} - \frac{75}{2} + 294 - 25$

$= \frac{-288-175-75+588-50}{2} = \frac{-588+588}{2} = 0$

$D = ab-h^2 = -36-\frac{25}{4} = \frac{-25-144}{4} = \frac{-169}{4}$

Since, $\Delta=0$ and $D\neq 0$, (i) represents pair of intersecting straight lines.

Let, (α,β) be the point of intersection

$\therefore 6\alpha-\frac{5}{2}\beta+7=0$ ---(ii)

$-\frac{5}{2}\alpha-6\beta+\frac{5}{2}=0$ ---(iii)

~~$\alpha = \frac{\frac{5}{2}\beta-7}{6}$, $\alpha = \frac{6\beta-\frac{5}{2}}{}$~~

Solving (ii) and (iii), we have,

$$\alpha = -\frac{11}{13} \text{ and } \beta = \frac{10}{13}$$

$\frac{\frac{5}{2}\beta-7}{6} = \frac{6\beta-\frac{5}{2}}{-\frac{5}{2}}$

$-\frac{25}{4}\beta+\frac{35}{2} = 36\beta-15$

or, $36\beta+\frac{25}{4}\beta = \frac{35}{2}+15$

$\Rightarrow \beta\times\frac{169}{4} = \frac{65}{2}$

or, $\beta = \frac{2\times 65}{169}$

$= \frac{130}{169} = \frac{10}{13}$

$\alpha = \frac{\frac{5}{2}\beta-7}{6}$

$= \frac{\frac{5}{2}\times\frac{10}{13}-7}{6}$

$= \frac{50-182}{26\times 6}$

$= -\frac{11}{13}$

Shifting origin to the point of intersection $\left(-\frac{11}{13},\frac{10}{13}\right)$, the eq^n (i) transfered to

$$6x'^2-5x'y'-6y'^2=0 \quad \text{---(iv)}$$

Let us now rotate the axes through an angle θ to remove $x'y'$ term

$\therefore \tan 2\theta = \frac{2h}{a-b} = \frac{-5}{12}$

$\therefore \theta = \frac{1}{2}\tan^{-1}\left(\frac{-5}{12}\right)$

$\therefore$ The eq^n (iv) becomes, $aX^2+bY^2=0$ ---(v)

$\therefore$ By the law of invarients,

$a+b=0$

$ab = -36-\frac{25}{4} = -\frac{169}{4}$

$a^2 = \frac{169}{4}$

$a = \pm\frac{13}{2}$ $\therefore b = -\frac{13}{2}$

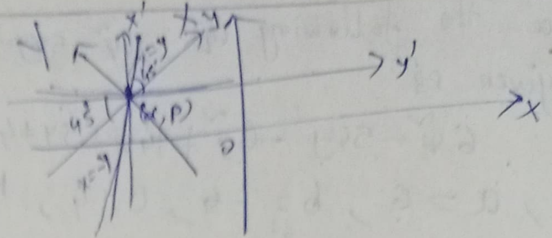
or, $b=\frac{13}{2}$

$\frac{13}{2}x^2 - \frac{13}{2}y^2 = 0$

or, $(x+y)(x-y) = 0$

~~$\therefore x = y, x = -y$~~

or, $y = x$, $y = -x$



5) Reduce the following eq^n to it's canonical form $x^2+2xy+y^2-4x-4y+3=0$

=> The given eq^n,

$$x^2+2xy+y^2-4x-4y+3=0 \quad \text{---(i)}$$

Here, $a=1$, $b=1$, $c=3$, $h=1$, $g=-2$, $f=-2$

$\therefore \Delta = abc + 2fgh - bg^2 - af^2 - ch^2$

$= \cancel{3} + 8 - 4 - 4 - \cancel{3} = 0$

$\therefore D = ab - h^2 = 1 - 1 = 0$

Since, $D = 0$ and $\Delta = 0$

$\therefore$ (i) represents pair of parallal straight lines.

~~$\therefore$ Let, (α,β)~~

$\therefore$ The ~~given~~ eq^n (i) ~~can be written as~~,

$$x^2+2xy+y^2 = 4x+4y-3$$

or, $(x+y)^2 = 4x+4y-3$

or, $(x+y+\lambda)^2 = 4x+4y-3+\lambda^2+2\lambda(x+y)$

$= x(4+2\lambda) + y(4+2\lambda) + (\lambda^2-3)$ ---(ii)

λ is chosen such that the straight lines $x+y+\lambda=0$ ---(iii) and $x(4+2\lambda)+y(4+2\lambda)+(\lambda^2-3)=0$ ---(iv) are mutually perpendicular.

$\therefore -1\left(-\frac{4+2\lambda}{4+2\lambda}\right) = -1$

Let us rotate the axes through an ~~agn~~ angle θ.

$\therefore$ By law of rotation we have,

$$x = X\cos\theta - Y\sin\theta$$
$$y = X\sin\theta + Y\cos\theta$$

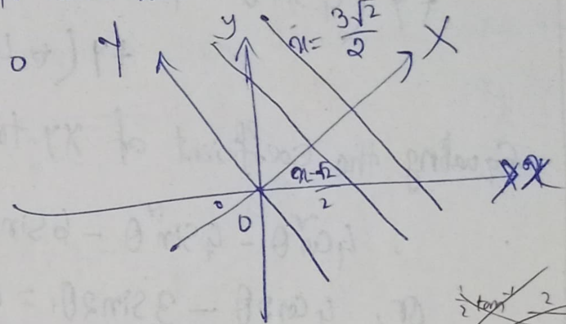
$\therefore$ from (i),

$$(X\cos\theta - Y\sin\theta)^2 + 2(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta) + (X\sin\theta + Y\cos\theta)^2 - 4(X\cos\theta - Y\sin\theta) - 4(X\sin\theta + Y\cos\theta) + 3 = 0$$

or, $$(\cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta)X^2 + (-2\sin\theta\cos\theta + 2\sin\theta\cos\theta - 2\sin^2\theta + 2\sin\theta\cos\theta)XY + (\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta)Y^2 + (-4\cos\theta - 4\sin\theta)X + (4\sin\theta - 4\cos\theta)Y + 3 = 0 \quad \text{—(ii)}$$

Equating the coefficients of XY to zero we have,

$$\cos^2\theta - \sin^2\theta = 0$$
$$\sin\theta = \cos\theta$$
$$\therefore \theta = \frac{\pi}{4}$$



$\therefore$ from (ii),

$$2X^2 - 4\sqrt{2}X + 3 = 0$$
$$X = \frac{4\sqrt{2} \pm \sqrt{32 - 24}}{4}$$
$$= \frac{4\sqrt{2} \pm 2\sqrt{2}}{4} = \frac{2\sqrt{2} \pm \sqrt{2}}{2}$$
$$= \sqrt{2} \pm \frac{1}{\sqrt{2}}$$

$$\therefore X = \frac{3\sqrt{2}}{2}, \quad X = \frac{\sqrt{2}}{2}$$

6) Reduce the eq^n $4x^2 + 4xy + y^2 - 4x - 2y + a = 0$ to it's canonical form and determine it's nature for different value's of a.

$\Rightarrow$ The given eq^n, $4x^2 + 4xy + y^2 - 4x - 2y + a = 0$ —(i)

Here, $a = 4$, $b = 1$, $c = a$, $f = -1$, $g = -2$, $h = 2$

$\therefore \Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$D = ab - h^2 = 4 - 4 = 0$

Since, $\Delta = 0$ and $D = 0$, (i) represents pair of parallel straight lines.

Let us rotate the axes through an angle θ to remove the xy term.

$\therefore$ By ~~of~~ law of rotation,

$$x = X\cos\theta - Y\sin\theta$$

and $y = X\sin\theta + Y\cos\theta$

$\therefore$ from (i),

$$4(X\cos\theta - Y\sin\theta)^2 + 4(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta)$$
$$+(X\sin\theta + Y\cos\theta)^2 - 4(X\cos\theta - Y\sin\theta) - 2(X\sin\theta + Y\cos\theta) + a = 0$$

or, $X^2(4\cos^2\theta + 4\sin\theta\cos\theta + \sin^2\theta) + XY(-8\sin\theta\cos\theta + 4\cos^2\theta - 4\sin^2\theta + 2\sin\theta\cos\theta)$

$+Y^2(4\sin^2\theta - 4\sin\theta\cos\theta + \cos^2\theta) + X(-4\cos\theta - 2\sin\theta)$

$+Y(4\sin\theta - 2\cos\theta) + a = 0$ ...(ii)

Equating the coefficient of XY to zero, we have,

$$4\cos^2\theta - 4\sin^2\theta - 6\sin\theta\cos\theta = 0$$

or, $4\cos 2\theta - 3\sin 2\theta = 0$

or, $4\cos 2\theta = 3\sin 2\theta$

or, $\tan 2\theta = \frac{4}{3}$

or, $\theta = \frac{1}{2}\tan^{-1}\frac{4}{3}$

$\therefore \sin 2\theta = \frac{4}{5}$, $\cos 2\theta = \frac{3}{5} \Rightarrow 2\cos^2\theta - 1 = \frac{3}{5}$ $\therefore \sin\theta = \frac{1}{\sqrt{5}}$

$\therefore \cos^2\theta = \frac{4}{5}$

$\cos\theta = \frac{2}{\sqrt{5}}$

$\therefore$ from (ii)

$$\left(2\sin 2\theta + \frac{3}{2}\cos 2\theta + \frac{5}{2}\right)X^2 + (-4\cos\theta - 2\sin\theta)X + a = 0$$

or, $\left(4\cdot\frac{4}{5} + 2\cdot\frac{4}{5} + \frac{1}{5}\right)X^2 + \left(-4\frac{2}{\sqrt{5}} - 2\frac{1}{\sqrt{5}}\right)X + a = 0$

or, $\frac{16+8+1}{5}X^2 + \frac{-8-2}{\sqrt{5}}X + a = 0$

or, $5X^2 - 2\sqrt{5}X + a = 0$

$$\therefore X = \frac{2\sqrt{5} \pm \sqrt{20 - 20a}}{10} = \frac{\sqrt{5} \pm \sqrt{5 - 5a}}{5}$$

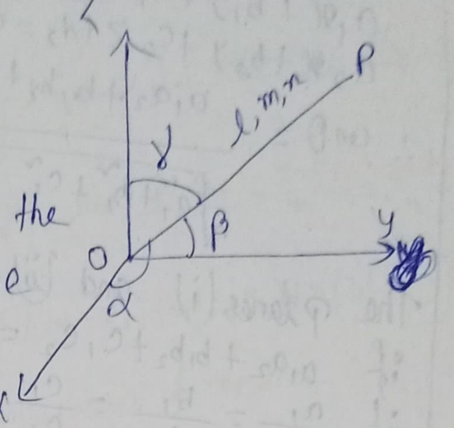
$$= \frac{\sqrt{5} \pm \sqrt{5}\sqrt{1-a}}{5} = \frac{1 \pm \sqrt{1-a}}{\sqrt{5}}$$

Case I :- when $a=1$, (i) represents pair of coincidant straight lines

Case II :- for $a<1$, (i) represents pair of parallel straight lines (non coincident)

Case III :- for $a>1$, (i) represents pair of imaginary straight lines.

Direction Cosines



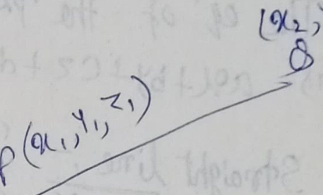
$l=\cos\alpha$, $m=\cos\beta$, $n=\cos\gamma$

(i) **Direction ratio :-** If a, b, c be the direction ratio's and l, m, n be the direction cosines then

$$\frac{l}{a}=\frac{m}{b}=\frac{n}{c}$$

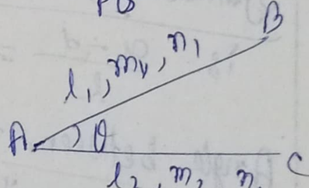
(ii) If a, b, c be the direction ratio's then direction cosines are

$$\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}$$



(iii) direction ratio's of PQ are $x_2-x_1, y_2-y_1, z_2-z_1$

and direction cosines are $\frac{x_2-x_1}{PQ}$; $\frac{y_2-y_1}{PQ}$, $\frac{z_2-z_1}{PQ}$



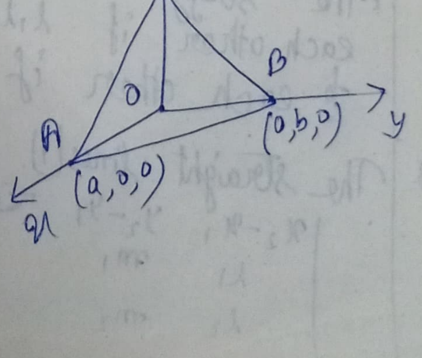
(iv) $$\cos\theta=\frac{l_1l_2+m_1m_2+n_1n_2}{\sqrt{l_1^2+m_1^2+n_1^2}\sqrt{l_2^2+m_2^2+n_2^2}}$$

(v) Condition of perpendicularity is $l_1l_2+m_1m_2+n_1n_2=0$

(vi) Condition of parallelism, $\frac{l_1}{l_2}=\frac{m_1}{m_2}=\frac{n_1}{n_2}$.

III Plane :-

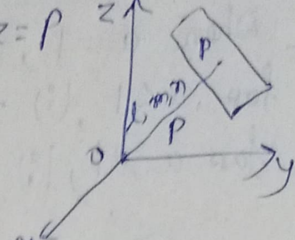
1) The general eq^n of plane is $ax+by+cz+d=0$, where a, b, c are the direction ratio's of the normal to the plane



(ii) **Intercept form of a Plane :-**

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

Normal form of a Plane :— $lx + my + nz = P$



Angle between two planes :—

$a_1x + b_1y + c_1z + d_1 = 0$ — (i)
$a_2x + b_2y + c_2z + d_2 = 0$ — (ii)

$$\therefore \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

The planes (i) and (ii) are perpendicular to each other if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ and parallel to each other if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

The eqⁿ of the plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + d' = 0$

Straight line :— In three dimension the general eqⁿ of straight line is $a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z + d_2$

Symmetric form :— The symmetric form of a straight line is $\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}$

Angle between twisted lines :—

$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$ — (i)

$\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$ — (ii)

$$\therefore \cos\theta = \frac{l_1l_2 + m_1m_2 + n_1n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2}\sqrt{l_2^2 + m_2^2 + n_2^2}}$$

The straight lines (i) and (ii) are perpendicular to each other if $l_1l_2 + m_1m_2 + n_1n_2 = 0$ and parallel to each other if $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

The straight lines (i) and (ii) are co-planer if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

1 Consider the plane $ax+by+cz+d=0$ — (i) and the straight line

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad \text{— (ii)}$$

The straight line (ii) is perpendicular to the plane (i) if $\frac{a}{l}=\frac{b}{m}=\frac{c}{n}$ and parallel if $al+bm+cn=0$

~~A result:-~~

$$\cancel{d^2 = \{(\alpha-x_1)^2+(\beta-y_1)^2+(\gamma-z_1)^2 - \{l(\alpha-x_1)+m(\beta-y_1)+n(\gamma-z_1)\}^2}$$

A result:-

The perpendicular distance of the straight line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ from the point (α,β,γ) is given by,

$$d^2 = \{(\alpha-x_1)^2+(\beta-y_1)^2+(\gamma-z_1)^2\} - \{l(\alpha-x_1)+m(\beta-y_1)+n(\gamma-z_1)\}^2$$

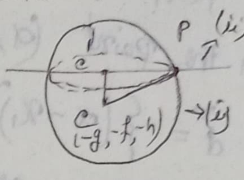
where, l,m,n are the actual direction cosines of the line.

Spheres

1) The eq^n of the sphere with centre at (x_1, y_1, z_1) and radius r is $(x-x_1)^2+(y-y_1)^2+(z-z_1)^2 = r^2$

2) The general eq^n of sphere is $x^2+y^2+z^2+2gx+2fy+2hz+c=0$. The radius of sphere is $\sqrt{g^2+f^2+h^2-c}$ and centre is at $(-g, -f, -h)$

3) The sphere $x^2+y^2+z^2+2gx+2fy+2hz+c=0$ —(i) and the plane $ax+by+cz+d=0$ —(ii) togetherly represents a circle.



[Note:- If C' coincides with C then the circle is called great circle.]

4) Let, $\left.\begin{array}{l} S \equiv x^2+y^2+z^2+2gx+2fy+2hz+c=0 \\ L \equiv ax+by+cz+d=0 \end{array}\right\}$ and —(A)

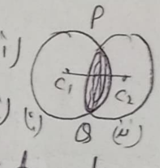
eq^n of the sphere containing the circle (A) is

$S+\lambda L=0$

$\Rightarrow (x^2+y^2+z^2+2gx+2fy+2hz+c)+\lambda(ax+by+cz+d)=0$

5) Let, $S_1 \equiv x^2+y^2+z^2+2g_1x+2f_1y+2h_1z+c_1=0$ —(i)

$S_2 \equiv x^2+y^2+z^2+2g_2x+2f_2y+2h_2z+c_2=0$ —(ii)



The eq^n of the plane on which the circle of intersection of the spheres (i) and (ii) lies, is $S_1 - S_2 = 0$

6) S_1 and S_2 cut each other orthogonaly if

$2g_1g_2+2f_1f_2+2h_1h_2 = c_1+c_2$

7) The eq^n of the ~~sh~~ sphere containing the circle of intersection of (i) and (ii) is $S_1+\lambda S_2=0$

8) The point $P(x_1, y_1, z_1)$ lies outside, inside or on the sphere S_1 if $x_1^2+y_1^2+z_1^2+2gx_1+2fy_1+2hz_1+c$ $>, <$ or $=0$.

9) Consider the sphere $x^2+y^2+z^2=r^2$ and the point $C'(a,b,c)$. The eq^n of the plane on which the circle with centre at C' lies is $a(x-a)+b(y-b)+c(z-c)=0$

10) If the plane $ax+by+cz+d=0$ (ii) touches the sphere $x^2+y^2+z^2+2gx+2fy+2hz+c=0$ (i) then the plane is called tangent plane.

The plane (ii) will be a tangent plane to the sphere (i) if the perpendicular distance of the plane from the centre of sphere (i) ~~equ~~ = radius of the sphere (i).

1) Find the eqⁿ of sphere passing through the points $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, $(0,0,c)$.

⇒ Let, the eqⁿ of the sphere be $x^2+y^2+z^2+2gx+2fy+2hz+c'=0$ — (i)

By the given condition,

$c'=0$ — (ii)

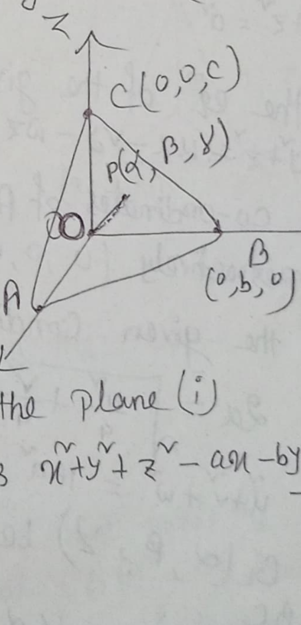
$a^2+2ga+c'=0$ — (iii) $\Rightarrow g=-\frac{a}{2}$

$b^2+2fb+c'=0$ — (iv) $\Rightarrow f=-\frac{b}{2}$

$c^2+2hc+c'=0$ — (v) $\Rightarrow h=-\frac{c}{2}$

∴ The required eqⁿ of the sphere $\boxed{x^2+y^2+z^2-ax-by-cz=0}$

2) A sphere of constant radius r passes through the origin O and cuts the axes in A, B, C. Prove that the locus of the foot of perpendicular from O ~~to~~ to the plane ~~ABC~~ ABC is given by $(x^2+y^2+z^2)^2(x^{-2}+y^{-2}+z^{-2})=4r^2$



⇒ Let, the eqⁿ of the plane ABC be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \text{ — (i)}$$

∴ The co-ordinates of A, B and C are respectively $(a,0,0)$, $(0,b,0)$, $(0,0,c)$.

Let, $P(\alpha,\beta,\gamma)$ be the foot of perpendicular ~~to~~ drawn from O upon the plane (i)

Now, the eqⁿ of the sphere OABC is $x^2+y^2+z^2-ax-by-cz=0$ — (ii).

∴ By the given condition,

$$r=\sqrt{\frac{a^2}{4}+\frac{b^2}{4}+\frac{c^2}{4}}$$

or, $a^2+b^2+c^2=4r^2$ — (iii)

Now, $OP=\sqrt{\alpha^2+\beta^2+\gamma^2}=\left|\frac{-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right|=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$

or, $\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)(\alpha^2+\beta^2+\gamma^2)=1$ — (iv)

Again, the direction ratio's of $\overline{OP}$ are $\alpha, \beta, \gamma = \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$

$\therefore \alpha = \frac{1}{a}, \beta = \frac{1}{b}, \gamma = \frac{1}{c}$

$\therefore a = \frac{1}{\alpha}, b = \frac{1}{\beta}, c = \frac{1}{\gamma}$

Putting the value's of a, b, c into (iii) and (iv) we have,

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = 4r^2 \quad \text{—(v)}$$

and $$(\alpha^2+\beta^2+\gamma^2)^2 = 1 \quad \text{—(vi)}$$

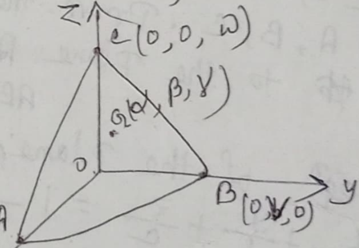
$\therefore$ from (v) and (vi)

$$\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}\right)(\alpha^2+\beta^2+\gamma^2)^2 = 4r^2$$

$\therefore$ The required locus,

$$(x^{-2}+y^{-2}+z^{-2})(x^2+y^2+z^2)^2 = 4r^2$$

3) A sphere of constant radius $2a$ passes through the origin O and meets the axes in A, B, C. Show that the locus of the centroid of the tetrahedron OABC is the sphere $x^2+y^2+z^2 = a^2$.



$\Rightarrow$ Let, the eqn of the given sphere is

$$x^2+y^2+z^2-ux-vy-wz=0 \quad \text{—(i)}$$

$\therefore$ The co-ordinates of A, B and C are respectively $(u,0,0)$, $(0,v,0)$ and $(0,0,w)$.

$\therefore$ By the given condition,

$$2a = \sqrt{\frac{u^2}{4} + \frac{v^2}{4} + \frac{w^2}{4}}$$

or, $u^2+v^2+w^2 = 16a^2$ —(ii)

Let, $G(\alpha, \beta, \gamma)$ be the centroid of the tetrahedral OABC

$\therefore \alpha = \frac{u}{4} \Rightarrow u = 4\alpha$

$\beta = \frac{v}{4} \Rightarrow v = 4\beta$

$\gamma = \frac{w}{4} \Rightarrow w = 4\gamma$

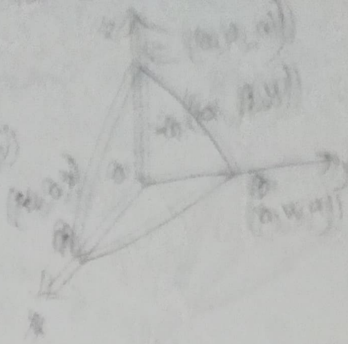
$\therefore$ from (ii),

$$16\alpha^2 + 16\beta^2 + 16\gamma^2 = 16a^2$$

or, $\alpha^2+\beta^2+\gamma^2 = a^2$

$\therefore$ The required locus, $(x^2+y^2+z^2) = a^2$

A sphere of radius k passes through the origin and meets the axes in A, B, C. Prove that the locus of the centroid of the triangle ABC is the sphere $9(x^2+y^2+z^2)=4k^2$.



Let, the eq^n of the sphere is,

$x^2+y^2+z^2-ux-vy-wz=0$ —— (i)

$\therefore k=\sqrt{\frac{u^2}{4}+\frac{v^2}{4}+\frac{w^2}{4}}$

or, $u^2+v^2+w^2=4k^2$ —— (ii)

Let, the co-ordinates of the centroid G is $((\alpha,\beta,\gamma))$ of the triangle ABC

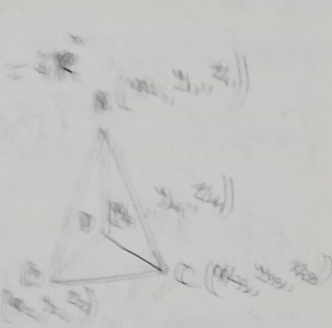
$\alpha=\frac{u}{3} \Rightarrow u=3\alpha$

$v=3\beta$,

$w=3\gamma$

$\therefore$ from (ii), $9(\alpha^2+\beta^2+\gamma^2)=4k^2$

$\therefore$ The required locus, $9(x^2+y^2+z^2)=4k^2$



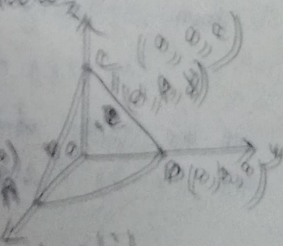
Volume of tetrahedron :-

Volume of tetrahedron ABCD

$$=\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

Note :- If D be the origin then the volume of the tetrahedron OABC is $\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

A variable sphere passes through the origin O and meets the axes in ABC so that the volume of the tetrahedral OABC is equal to a constant K. Show that the locus of the centre of the sphere is $4xyz=3K$



Let, the eq^n of the sphere be

$x^2+y^2+z^2-xa-yb-zc=0$ —— (i)

$\therefore$ The centre of (i) is $\left(\frac{a}{2},\frac{b}{2},\frac{c}{2}\right)$

Let, $C'(\alpha,\beta,\gamma)$ be the centre of the sphere (i)

$\therefore \alpha = \frac{a}{2}, \beta = \frac{b}{2}, \gamma = \frac{c}{2}$

$\therefore a = 2\alpha, b = 2\beta, c = 2\gamma$

$\therefore$ By the given condition,

$$\frac{1}{6}\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = K$$

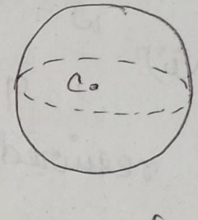
or, $\frac{1}{6} abc = K$

or, $\frac{4}{3}\alpha\beta\gamma = K$

or, $4\alpha\beta\gamma = 3K$

$\therefore$ The required locus, $\boxed{4xyz = 3K}$

6) Find the eq^n of the sphere having the circle $x^2+y^2+z^2=9$, $x+y+z+3=0$ as a great circle.



$\Rightarrow$ The given eq^n is

$$\left.\begin{aligned} x^2+y^2+z^2 &= 9 \quad \text{—(i)} \\ x+y+z+3 &= 0 \quad \text{—(ii)} \end{aligned}\right\} \text{—(A)}$$

The eq^n of the sphere containing the circle A is,

$x^2+y^2+z^2-9+\lambda(x+y+z+3)=0$

or, $x^2+y^2+z^2+x\lambda+y\lambda+z\lambda+3\lambda-9=0$ —(iii)

The centre of the sphere (iii) is $C\left(-\frac{\lambda}{2}, -\frac{\lambda}{2}, -\frac{\lambda}{2}\right)$

Since, A is the great circle of (iii)

$\therefore$ C lies on the plane (ii).

$\therefore -\frac{\lambda}{2}-\frac{\lambda}{2}-\frac{\lambda}{2}+3=0$

or, $-\frac{3\lambda}{2} = -3$

or, $\lambda = 2$

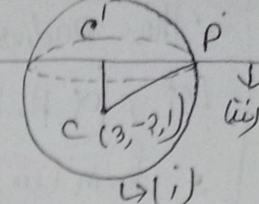
$\therefore$ ~~from (iii)~~ Putting the value of λ into (iii),

$x^2+y^2+z^2+2x+2y+2z-3=0$

Find the centre ~~of the~~ and radius of the circle $(x-3)^2+(y+2)^2+(z-1)^2=100$ and $2x-2y-z+9=0$.

The given eq^n is,

$$\left.\begin{array}{l}(x-3)^2+(y+2)^2+(z-1)^2=100 \quad \text{--- (i)} \\ 2x-2y-z+9=0 \quad \text{--- (ii)}\end{array}\right\} \text{--- (A)}$$



The centre of the sphere (i) is $c(3,-2,1)$.

Let, c' be the centre of the circle A and $c'P$ be it's radius

from figure we have, $CP=10$

$$cc' = \left| \frac{2\cdot 3-2(-2)-1\cdot 1+9}{\sqrt{2^2+2^2+1^2}} \right| = 6$$

$\therefore$ The radius of the circle is $c'P = \sqrt{cP^2-cc'^2}$

$$= \sqrt{100-36} = 8 \text{ unit}$$

clearly, cc' is the normal to the plane (ii).

$\therefore$ Direction ratio's of cc' is $2, -2, -1$

$\therefore$ The eq^n of the straight line cc' is $\frac{x-3}{2} = \frac{y+2}{-2} = \frac{z-1}{-1}$ --- (iii)

Let, the co-ordinates of the point c' be $(2r+3, -2r-2, -r+1)$

Since, c' lies on the plane (ii).

$\therefore \; 2(2r+3)-2(-2r-2)-(-r+1)+9=0$

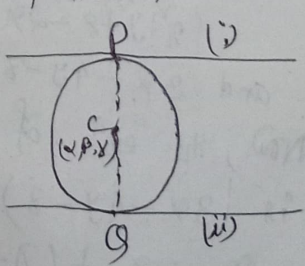
or, $4r+6+4r+4+r-1+9=0$

or, $9r+18=0$

or, $r=-2$

$\therefore$ The centre is $(-1, 2, 3)$

8) If a sphere touches the planes $2x+3y-6z+14=0$ and $2x+3y-6z+42=0$ and if it's centre lies on the straight line $2x+z=0$, $y=0$, find the eq^n of the sphere



$\Rightarrow$ The given eq^n's are,

$2x+3y-6z+14=0$ --- (i)

$2x+3y-6z+42=0$ --- (ii)

$2x+z=0$, $y=0$ --- (iii)

By the given condition, the diameter of the sphere is PCQ

$$= \frac{42-14}{\sqrt{2^2+3^2+6^2}}$$

$$= \frac{28}{7} = 4$$

$\therefore$ The radius of the sphere, $\overline{CP} = 2$

Let, (α, β, γ) be the centre of the sphere

$\therefore$ from (iii),

$$\left.\begin{array}{l} 2\alpha + \gamma = 0 \\ \beta = 0 \end{array}\right\} \text{—(iv)}$$

Again we have,

$$CP = \left| \frac{2\cdot\alpha + 3\cdot 0 - 6\cdot\gamma + 14}{\sqrt{4+9+36}} \right| = 2$$

~~or, $2\alpha - 6\gamma + 28 = 0$ — (v)~~

or, $2\alpha - 6\gamma + 14 = -2\times 7$

or $2\alpha - 6\gamma + 28 = 0$

or, $2\alpha = 6\gamma - 28$

$\therefore$ $6\gamma - 28 = -\gamma$

$7\gamma = 28$

$\gamma = 4$

$\therefore$ $\alpha = -\frac{\gamma}{2} = -2$

$\therefore$ $\alpha = -2, \beta = 0, \gamma = 4$

$\therefore$ The required eq^n is $(x+2)^2 + y^2 + (z-4)^2 = 4$

9) Show that only one tangent plane can be drawn to the sphere $x^2+y^2+z^2-2x+6y+2z+8=0$ through the straight line $3x-4y-8=0=y-3z+2$. Find the eq^n of the plane

$\Rightarrow$ the given eq^ns are,

$$x^2+y^2+z^2-2x+6y+2z+8=0 \text{ —(i)}$$

$$\text{and } 3x-4y-8=0=y-3z+2 \text{ —(ii)}$$

Now, the eq^n of the plane containing the straight line (ii),

is $(3x-4y-8) + \lambda(y-3z+2) = 0$

or, $3x + (\lambda-4)y - 3\lambda z + (2\lambda - 8) = 0$ —(iii)

Since, (iii) touches (i)

$\therefore$ The perpendicular distance of (iii) from the centre $(1,-3,-1)$ of the sphere (i) is equal to the radius of the sphere

$\therefore \left| \dfrac{3\cdot 1 + (\lambda-4)(-3) - 3\lambda(-1) + 2\lambda - 8}{\sqrt{3^2 + (\lambda-4)^2 + (3\lambda)^2}} \right| = \sqrt{1^2+3^2+1^2-8} = \sqrt{3}$

or, $\left| \dfrac{3 - \cancel{3\lambda} + 12 + \cancel{3\lambda} + 2\lambda - 8}{\sqrt{9+\lambda^2+16-8\lambda+9\lambda^2}} \right| = \sqrt{3}$

or, $(2\lambda+7)^2 = 3(10\lambda^2 - 8\lambda + 25)$

or, $4\lambda^2 + 49 + 28\lambda = 30\lambda^2 - 24\lambda + 75$

or, $26\lambda^2 - 52\lambda + 26 = 0$

or, $\lambda^2 - 2\lambda + 1 = 0$

or, $(\lambda - 1)^2 = 0$

or, $\lambda = 1, 1$

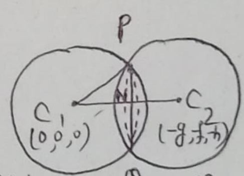
$\therefore$ Only one tangent plane can be drawn through the given straight line.

Putting the value of λ into (iii), the eq^n of the plane is

$3x - 3y - 3z \cancel{+} - 6 = 0$

or, $x - y - z - 2 = 0$

10) Two spheres of radii r_1 and r_2 cut orthogonally. Prove that the radius of their common circle is $\dfrac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$



$\Rightarrow$ Let, the eq^n of the spheres be

$x^2+y^2+z^2 = r_1^2$ — (i)

$x^2+y^2+z^2+2gx+2fy+2hz+c=0$ — (ii)

By the given condition $r_2 = \sqrt{g^2+f^2+h^2-c}$

Now, the eq^n of the plane on which the common circle lies is ~~2gx+2fy~~ $S_1 - S_2 = 0$

$2gx+2fy+2hz+c+r_1^2 = 0$ — (iii)

or, $2gx+2fy$

Now, $C_1N = \dfrac{c+r_1^2}{\sqrt{4g^2+4f^2+4h^2}} = \dfrac{c+r_1^2}{\sqrt{4(r_2^2+c)}}$

$\therefore PN^2 = C_1P^2 - C_1N^2$

$= r_1^2 - \dfrac{(c+r_1^2)^2}{4(r_2^2+c)} = \dfrac{4r_2^2 r_1^2 + 4cr_1^2 - r_1^4 - c^2 - 2cr_1^2}{4(r_2^2+c)}$

$= \dfrac{4r_2^2 r_1^2 - r_1^4 + 2cr_1^2}{4(r_2^2+c)} = \dfrac{4r_2^2 r_1^2 - r_1^4 + 2r_1^4}{4(r_2^2+r_1^2)}$

$\because$ (i) and (ii) cut each other orthogonally,

$\therefore 2g_1g_2 + 2f_1f_2 + 2h_1h_2 = c_1 + c_2$

$\therefore \quad c - r_1^2 = 0$

$\quad c = r_1^2$

$$\therefore PN^2 = \cancel{\frac{4r_1^2r_2^2 + r_1^4}{4(r_2^2+r_1^2)}} = \cancel{r_1^2(4r_2^2+r_1^2)}$$

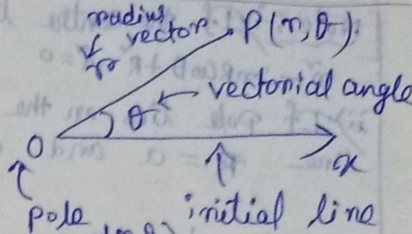
$$= C_1P^2 - C_1N^2 = r_1^2 - \frac{(c+r_1^2)^2}{4(r_2^2+c)} \quad [\because c = r_1^2]$$

$$= r_1^2 - \frac{(2r_1^2)^2}{4(r_2^2+r_1^2)} = r_1^2 - \frac{r_1^4}{r_1^2+r_2^2}$$

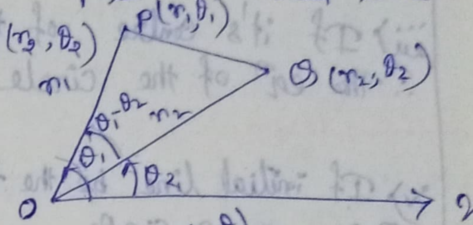
$$= \frac{\cancel{r_1^4} + r_1^2r_2^2 - \cancel{r_1^4}}{r_1^2+r_2^2} = \frac{r_1^2r_2^2}{r_1^2+r_2^2}$$

$$\therefore PN = \frac{r_1r_2}{\sqrt{r_1^2+r_2^2}}$$

Polar eqⁿ

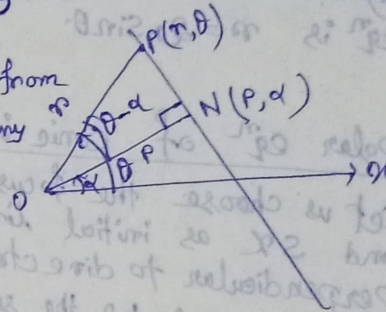


Distance between two points :-



The distant between the points $P(r_1,\theta_1)$ and $Q(r_2,\theta_2)$ is $PQ = \sqrt{r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)}$

Polar eqⁿ of a straight line :-



Let, $N(P,\alpha)$ be the foot of perpendicular drawn from Pole upon the straight line. Let, $P(r,\theta)$ be any point on the straight line.

$\therefore$ from ΔOPN we have,

$$\cos(\theta-\alpha) = \frac{P}{r}$$

$$\therefore \boxed{r\cos(\theta-\alpha) = P}$$

This is the Polar eqⁿ of a straight line.

[Note :- i) The eqⁿ of line which is perpendicular to the initial line is $r\cos\theta = P$]

ii) The eqⁿ of line parallal to the initial line is $r\sin\theta = P$

iii) The eqⁿ of the straight line parallal to the straight line $r\cos(\theta-\alpha)=P$ is $r\cos(\theta-\alpha) = P'$

iv) The eqⁿ of the straight line perpendicular to the straight line $r\cos(\theta-\alpha) = P$ is $r\cos(\theta-\alpha') = P'$, where $\alpha \sim \alpha' = \frac{\pi}{2}$

v) $r\cos(\theta-\alpha)=P$

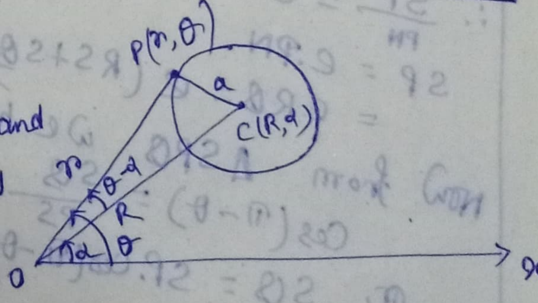
or, $r\cos\theta\cos\alpha + r\sin\theta\sin\alpha = P$

or, $r\cos\theta\frac{\cos\alpha}{P} + r\sin\theta\frac{\sin\alpha}{P} = 1 \Rightarrow \frac{1}{r} = \frac{\cos\alpha}{P}\cos\theta + \frac{\sin\alpha}{P}\sin\theta$

or, $\boxed{\frac{1}{r} = A\cos\theta + B\sin\theta}$

This is the general eqⁿ of straight line in Polar co-ordinate system.

Polar eqⁿ of a circle :-



Let, $C(R,\alpha)$ be the centre of the circle and a be it's radius. Let, $P(r,\theta)$ be any point on the circle.

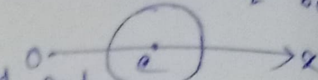
from ΔOPC we have,

$$PC^2 = r^2+R^2-2rR\cos(\theta-\alpha)$$

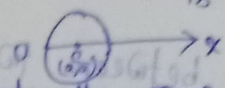
$\therefore$ or, $r^2+R^2-2rR\cos(\theta-\alpha)-a^2 = 0$

This is the general polar eqⁿ of circle.

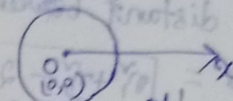
[Note :- i) If the centre lies on the initial line then $\alpha = 0$ and the eq^n becomes
$r^2 - 2rR\cos\theta + R^2 - a^2 = 0$



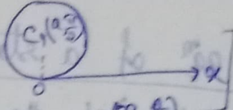
ii) If pole lies on the circumference of the circle and centre lies on initial line $R = a$ and $\alpha = 0$. Therefore, the eq^n of the circle is $\boxed{r = 2a\cos\theta}$



iii) If it's centre of the circle be the pole then $R = 0$, $\alpha = 0$
$\therefore$ The eq^n of the circle is $\boxed{r = a}$



iv) If initial line be the tangent to the circle at pole then it's polar eq^n is $r = 2a\sin\theta$



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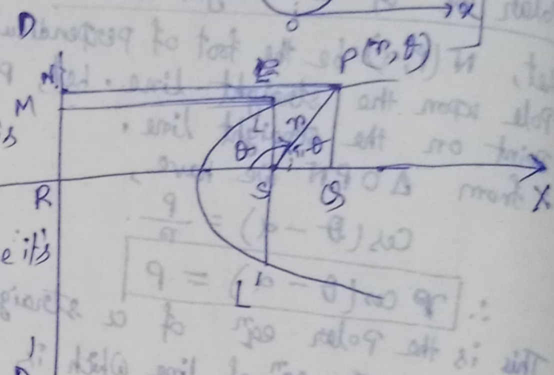
Polar eq^n of conic :-



Let us choose the focus S as pole and SX as initial line, which is perpendicular to directrix DD'.
Let, $SL = l$ be the semilatus rectum of the conic, and e be it's eccentricity. Let, M be the foot of perpendicular drawn from L upon the directrix.

$\therefore$ We have, $\frac{SL}{ML} = e$

$SL = e \cdot ML$

or, $l = e \cdot SR$ —— (i)

Let, $P(r, \theta)$ be any point on the conic, PN perpendicular to DD' and $PQ \perp SX$

$\therefore \frac{SP}{PN} = e$

$SP = e \cdot PN$
$= eRQ = e(RS + SQ) = e.RS + e.SQ$ —— (ii)

Now from $\triangle SPQ$ we have,
$\cos(\pi - \theta) = \frac{SQ}{PS}$

or, $SQ = SP.\cos(\pi - \theta) = -r\cos\theta$

$\therefore$ from (ii), $r = e.RS + e.r\cos\theta$
$r = e \cdot \frac{l}{e} - e.r\cos\theta$ [from (i)]
$= l - e\,r\cos\theta$

$r + e\,r\cos\theta = l$

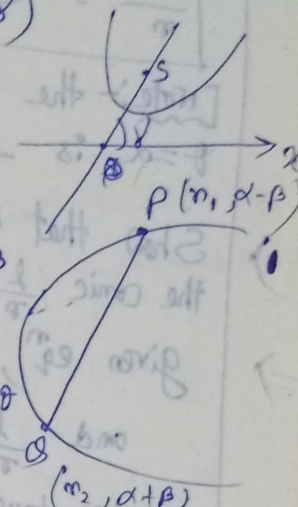
or, $1 + e\cos\theta = \frac{l}{r}$

or, $\boxed{\frac{l}{r} = 1 + e\cos\theta}$ —— (iii) This is the polar eq^n of a conic.

[Note :- i) The conic (ii) represents ellipse, parabola or hyperbola according as $e <, =$ or > 1]

ii) If S be the pole and SX' be the initial line then the eq^n of conic is $\frac{l}{r} = 1 - e\cos\theta$

iii) If the axis of the conic be inclined at an angle γ to the initial line then the eq^n of the conic is $\frac{l}{r} = 1 + e\cos(\theta - \gamma)$



1) Polar Eq^n of chord of a conic :-

Let, the eq^n of the conic be $\frac{l}{r} = 1 + e\cos\theta$ —— (i)

Let, P and Q be two points on the conic whose vectorial angles are respectively $\alpha - \beta$ and $\alpha + \beta$

Let, r_1 and r_2 be the radius vectors of P and Q respectively.

Let, the eq^n of the chord PQ be $\frac{l}{r} = A\cos\theta + B\sin\theta$ —— (ii)

Since, P lies on both in (i) and (ii).

∴ we have,

$$\frac{l}{r_1} = 1 + e\cos(\alpha - \beta) \quad \text{and} \quad \frac{l}{r_1} = A\cos(\alpha-\beta) + B\sin(\alpha-\beta)$$

$$\therefore A\cos(\alpha-\beta) + B\sin(\alpha-\beta) = 1 + e\cos(\alpha-\beta)$$

or, $(A - e)\cos(\alpha-\beta) + B\sin(\alpha-\beta) - 1 = 0$ —— (iii)

Similarly, since Q lies on both (i) and (ii), we have,

$(A-e)\cos(\alpha+\beta) + B\sin(\alpha+\beta) - 1 = 0$ —— (iv)

Solving (iii) and (iv) for $A-e$ and B by means of cross multiplication we have,

$$\frac{A-e}{\sin(\alpha+\beta) - \sin(\alpha-\beta)} = \frac{B}{\cos(\alpha-\beta) - \cos(\alpha+\beta)} = \frac{1}{\cos(\alpha-\beta)\sin(\alpha+\beta) - \cos(\alpha+\beta)\sin(\alpha-\beta)}$$

or, $\frac{A-e}{2\sin\beta\cos\alpha} = \frac{B}{2\sin\beta\sin\alpha} = \frac{1}{2\sin\beta\cos\beta}$

or, $A = e + \cos\alpha\sec\beta$ and $B = \sin\alpha\sec\beta$

∴ from (ii) the eq^n of the chord PQ is

$\frac{l}{r} = (e + \cos\alpha\sec\beta)\cos\theta + \sin\alpha\sec\beta\sin\theta$ —— (v)

2) Eq^n of tangent to a conic :-

Consider the conic $\frac{l}{r} = 1 + e\cos\theta$ —— (i)

we have, the eq^n of the chord joining the points P and Q whose vectorial angle is respectively $\alpha - \beta$ and $\alpha + \beta$ is $\frac{l}{r} = (e + \cos\alpha\sec\beta)\cos\theta + \sin\alpha\sec\beta\sin\theta$ —— (ii)

Now, the chord PQ will be tangent at P to the conic if $\beta = 0$.

$\therefore$ The eq^n of tangent at $\theta = \alpha$ to the conic (i) is

$\frac{l}{r} = (e+\cos\alpha)\cos\theta + \sin\alpha\sin\theta$

$= e\cos\theta + \cos\theta\cos\alpha\cos\theta + \sin\alpha\sin\theta$

$$\boxed{\frac{l}{r} = e\cos\theta + \cos(\theta-\alpha)}$$

[Note:- the eq^n of tangent to the conic $\frac{l}{r} = 1+e\cos(\theta-\gamma)$ is at $\theta=\alpha$ is $\frac{l}{r} = e\cos(\theta-\gamma) + \cos(\theta-\alpha)$]

1) Show that the straight line $\frac{l}{r} = A\cos\theta + B\sin\theta$ touches the conic $\frac{l}{r} = 1+e\cos\theta$ if $(A-e)^2 + B^2 = 1$

$\Rightarrow$ given eq^n, $\frac{l}{r} = A\cos\theta + B\sin\theta$ —— (i)

and $\frac{l}{r} = 1+e\cos\theta$ —— (ii)

Let, (i) touches (ii) at $\theta = \alpha$

Now we have the eq^n of tangent at $\theta=\alpha$ to the conic (ii) ~~—— (iii)~~

$\cancel{\frac{l}{r} = (e+\cos\alpha)\cos\theta + \sin\alpha\sin\theta}$

$\frac{l}{r} = e\cos\theta + \cos(\theta-\alpha)$ + $\sin\alpha\sin\theta$ —— (iii)

or, $\frac{l}{r} = (e+\cos\alpha)\cos\theta + \sin\alpha\sin\theta$

Since, (i) and (iii) are identical

$\therefore \frac{l}{l} = \frac{A}{e+\cos\alpha} = \frac{B}{\sin\alpha}$ —— (iv)

from (iv), $A = e+\cos\alpha$ and $\sin\alpha = B$

$\Rightarrow \cos\alpha = A-e$

we have, $\sin^2\alpha + \cos^2\alpha = 1$

$\therefore (A-e)^2 + B^2 = 1$ (Proved)

2) Show that the condition that the straight line $\frac{1}{r} = a\cos\theta + b\sin\theta$ may touch the conic $\frac{l}{r} = 1-e\cos\theta$ is $(al+e)^2 + b^2l^2 = 1$

$\Rightarrow$ given eq^n, $\frac{1}{r} = a\cos\theta + b\sin\theta$ —— (i) and $\frac{l}{r} = 1-e\cos\theta$ —— (ii)

Let the straight line (i) touches the conic (ii) at $\theta=\alpha$.

Now we have the eq^n of tangent at $\theta=\alpha$ to the conic (ii)

$\frac{l}{r} = -e\cos\theta + \cos(\theta-\alpha)$ + $\sin\alpha\sin\theta$ —— (iii)

or, $\frac{l}{r} = (\cos\alpha - e)\cos\theta + \sin\alpha\sin\theta$

Since, (i) and (iii) are identicles, compairing the coefficients we have,

$l\frac{1}{l} = \frac{a}{\cos\alpha - e} = \frac{b}{\sin\alpha}$ —— (iv)

from (iv),

$al = \cos\alpha - e$ and $\sin\alpha = bl$

$\Rightarrow \cos\alpha = al + e$

$\therefore$ we have,

$\sin^2\alpha + \cos^2\alpha = 1$

$\therefore (al+e)^2 + b^2l^2 = 1$ (Proved)

3) Show that the condition that the straight line $\frac{l}{r} = A\cos\theta + B\sin\theta$ may be a tangent to the conic $\frac{l}{r} = 1 + e\cos(\theta - \gamma)$ is $A^2 + B^2 - 2e(A\cos\gamma + B\sin\gamma) + e^2 = 1$

$\Rightarrow$ given that, $\frac{l}{r} = A\cos\theta + B\sin\theta$ —(i) and $\frac{l}{r} = 1 + e\cos(\theta - \gamma)$ —(ii)

Let, the straight line (i) touches the conic (ii) at $\theta = \alpha$.

Now we have the eqn of tangent at $\theta = \alpha$ to the conic (ii) is

$\frac{l}{r} = e\cos(\theta - \gamma) + \cos(\theta - \alpha)$

or, $\frac{l}{r} = (e\cos\gamma + \cos\alpha)\cos\theta + (e\sin\gamma + \sin\alpha)\sin\theta$ —(iii)

Since, (i) and (iii) are identicle, compairing the coefficients

$\frac{l}{l} = \frac{A}{e\cos\gamma + \cos\alpha} = \frac{B}{e\sin\gamma + \sin\alpha}$ —— (iv)

$\therefore A = e\cos\gamma + \cos\alpha$, $B = e\sin\gamma + \sin\alpha$

or, $\cos\alpha = A - e\cos\gamma$ or, $\sin\alpha = B - e\sin\gamma$

we have, $\sin^2\alpha + \cos^2\alpha = 1$

$(A - e\cos\gamma)^2 + (B - e\sin\gamma)^2 = 1$

$A^2 + e^2\cos^2\gamma - 2Ae\cos\gamma + B^2 + e^2\sin^2\gamma - 2Be\sin\gamma = 1$

or, $A^2 + B^2 - 2e(A\cos\gamma + B\sin\gamma) + e^2 = 1$ (Proved)

4) On the curve $\frac{r}{2} = \frac{l}{5 - 2\cos\theta}$, find the point with the smallest radius vector.

$\Rightarrow$ we have, $r = \frac{2l}{5 - 2\cos\theta}$

$\therefore r$ will be minimum when $5 - 2\cos\theta$ is maximum.

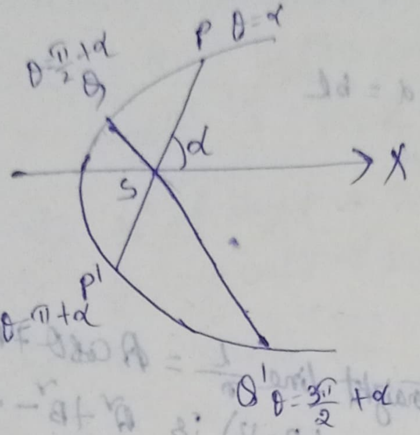
Now, $5 - 2\cos\theta$ is maximum when $\theta = \pi$.

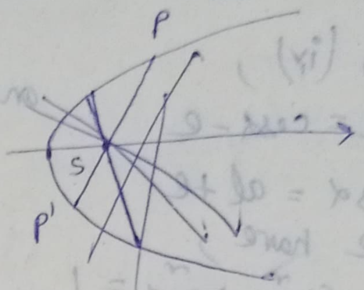
$\therefore r_{min} = \frac{2l}{7}$

$\therefore$ The Co-ordinate of the required point $\left(\frac{2l}{7}, \pi\right)$

[Note:- the radius vector for the above curve will be maximum if $\theta = 0$]

5) Prove that in a conic $\frac{l}{r} = 1 - e\cos\theta$, the sum of reciprocals of two perpendicular focal chords is constant.





Let, PSP' and QSQ' be two perpendicular focal chords.
Let, α be the vectorial angle of P.
∴ The vectorial angles of Q, P' and Q' are respectively $\frac{\pi}{2}+\alpha$, $\pi+\alpha$ and $\frac{3\pi}{2}+\alpha$.
from the given curve we have, $\cancel{r = \frac{1-e\cos\theta}{l}}$ —(i)

$$r = \frac{l}{1-e\cos\theta} \quad \text{—(i)}$$

$$\therefore SP = \frac{l}{1-e\cos\alpha}$$

$$\therefore SP' = \frac{l}{1-e\cos(\pi+\alpha)} = \frac{l}{1+e\cos\alpha}$$

$$\therefore PSP' = SP + SP' = \frac{l}{1-e\cos\alpha} + \frac{l}{1+e\cos\alpha}$$

$$= \frac{l(1+e\cos\alpha+1-e\cos\alpha)}{(1-e^2\cos^2\alpha)} = \frac{2l}{1-e^2\cos^2\alpha}$$

$$\therefore SQ = \frac{l}{1-e\cos\left(\frac{\pi}{2}+\alpha\right)} = \frac{l}{1+e\sin\alpha}$$

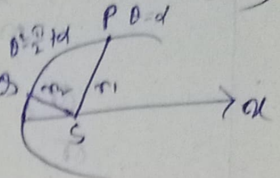
$$\therefore SQ' = \frac{l}{1-e\cos\left(\frac{3\pi}{2}+\alpha\right)} = \frac{l}{1-e\sin\alpha}$$

$$\therefore QSQ' = SQ + SQ' = \frac{l}{1+e\sin\alpha} + \frac{l}{1-e\sin\alpha} = \frac{2l}{1-e^2\sin^2\alpha}$$

$$\therefore \frac{1}{PSP'} + \frac{1}{QSQ'} = \frac{1-e^2\cos^2\alpha}{2l} + \frac{1-e^2\sin^2\alpha}{2l}$$

$$= \frac{2-e^2}{2l} = \text{constant.}$$

6) If r_1 and r_2 be two mutually perpendicular radius vectors of the ellipse $r^2 = \frac{b^2}{1-e^2\cos^2\theta}$, Prove that $\frac{1}{r_1^2}+\frac{1}{r_2^2} = \frac{1}{a^2}+\frac{1}{b^2}$, where $b^2 = a^2(1-e^2)$



$\Rightarrow$ Let, α be the vectorial angle of the point P

$\therefore$ vectorial angle of the point Q is $\frac{\pi}{2}+\alpha$

The given eq^n of the ellipse is,

$$r^2 = \frac{b^2}{1-e^2\cos^2\theta} \quad \text{---(i)}$$

$$\therefore r_1^2 = \frac{b^2}{1-e^2\cos^2\alpha} \text{ and } r_2^2 = \frac{b^2}{1-e^2\cos^2(\frac{\pi}{2}+\alpha)} = \frac{b^2}{1-e^2\sin^2\alpha}$$

$$\therefore \frac{1}{r_1^2}+\frac{1}{r_2^2} = \frac{1-e^2\cos^2\alpha}{b^2}+\frac{1-e^2\sin^2\alpha}{b^2} = \frac{2-e^2}{b^2}$$

$$= \frac{1+1-e^2}{b^2} = \frac{1+\frac{b^2}{a^2}}{b^2} = \frac{1}{b^2}+\frac{1}{a^2} \text{ (Proved)}$$

7) Find the nature of the following Conic and find it's semi-latus rectum $\frac{5}{r} = 2(1-\cos\theta)$

$\Rightarrow$ The given eq^n Can be written as, $\frac{5/2}{r} = 1-\cos\theta$

$\therefore l = \frac{5}{2}$ and $e = 1$

$\therefore$ The given curve is a parabola whose semi latus rectum is $\frac{5}{2}$.

8) Find the Point with greatest radius vector on the ellipse $\frac{21}{r} = 5-2\cos\theta$

$\Rightarrow$ The eq^n of the ellipse can be written as,

$$\frac{21}{r} = 5-2\cos\theta$$

$$\text{or, } \frac{1}{r} = \frac{5-2\cos\theta}{21}$$

$$\text{or, } r = \frac{21}{5-2\cos\theta}$$

Now, r will be maximum when $5-2\cos\theta$ is minimum.

i.e. when $\theta = 0$.

$$\therefore r_{max} = \frac{21}{5-2} = 7$$

$\therefore$ The required co-ordinate is $(7, 0)$

9) Write down the polar eq^n of the circle with radius a and centre at $(a, \frac{\pi}{2})$

$\Rightarrow$ The polar eq^n of the circle with centre at (R, α) and radius 'a' is $r^2 - 2rR\cos(\theta-\alpha)+R^2-a^2 = 0$

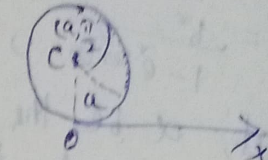
Here, $R = a$ and $\alpha = \frac{\pi}{2}$

∴ The required eqⁿ is,

$$r^2 - 2ar\cos\left(\theta - \frac{\pi}{2}\right) + a^2 - a^2 = 0$$

or, $r^2 - 2ar\sin\theta = 0$

or, $r = 2a\sin\theta$



10) If the straight line $r\cos(\theta-\alpha)=p$ touches the parabola $\frac{l}{r} = 1+\cos\theta$ then show that $p = \frac{l}{2}\sec\alpha$

⇒ The given eqⁿs are, $r\cos(\theta-\alpha) = p$ —(i)

and $\frac{l}{r} = 1+\cos\theta$ —(ii)

Let, the straight line (i) touches the parabola (ii) at $\theta = \alpha$

Now, the eqⁿ of tangent at $\theta = \beta$ to the parabola (ii) is,

$$\frac{l}{r} = \cos\theta + \cos(\theta-\beta)$$

$$= \cos\theta(1+\cos\beta) + \sin\beta\sin\theta \quad \text{—(iii)}$$

Now from (i) we have,

$$\frac{p}{r} = \cancel{\cos\beta\cos\theta + \sin\beta\sin\theta} \quad \text{—(iv)} \quad \cos\alpha\cos\theta + \sin\alpha\sin\theta$$

Since, (iii) and (iv) are identical,

$$\therefore \frac{l}{p} = \frac{1+\cos\beta}{\cos\beta} = \frac{\sin\alpha}{\sin\alpha} \quad \text{—(v)}$$

from P(v), $p = \frac{l\cos\alpha}{1+\cos\alpha} = \frac{l\cos\alpha(1-\cos\alpha)}{1-\cos^2\alpha} = \frac{l(\cos\alpha - \cos^2\alpha)}{\sin^2\alpha} =$

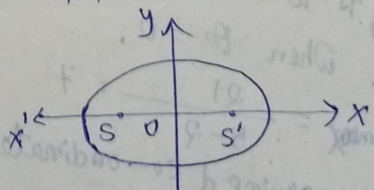
$= \cancel{\frac{l\cos\alpha}{2\cos^2\frac{\alpha}{2}}}$

11) Find the polar eqⁿ of the ellipse $\frac{x^2}{64} + \frac{y^2}{28} = 1$, if the pole be at it's right hand focus and the positive direction of x axis be the positive direction of polar axis.

⇒ Here, $a^2 = 64$, $b^2 = 28$

$$\therefore e = \sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\frac{28}{64}}$$

$$= \sqrt{1-\frac{7}{16}} = \frac{3}{4}$$



$$\therefore l = \frac{b^2}{a} = \frac{28}{8} = \frac{7}{2}$$

∴ The required eqⁿ is $\frac{l}{r} = 1 + e\cos\theta$

$\Rightarrow \frac{7}{2r} = 1 + \frac{3}{4}\cos\theta \Rightarrow \frac{14}{r} = 4 + 3\cos\theta$.

12) Show that the Point of intersection of the straight lines, $r\cos(\theta-\alpha)=P$ and $r\cos(\theta-\beta)=P$ is $\left(P\sec\frac{\alpha-\beta}{2}, \frac{\alpha+\beta}{2}\right)$

⇒ given eqⁿ's are, $r\cos(\theta-\alpha)=P$ ___ (i)

and $r\cos(\theta-\beta)=P$ ___ (ii)

from (i) and (ii),

$$r\cos(\theta-\alpha)=r\cos(\theta-\beta)$$
$$\cos(\theta-\alpha)=\cos(\theta-\beta)$$
$$\theta-\alpha=\pm(\theta-\beta)$$

$\theta-\alpha=-\theta+\beta$ (taking -ve sign)

or, $\theta=\frac{\alpha+\beta}{2}$

∴ from (i), $r=P\sec(\theta-\alpha)=P\sec\left(\frac{\alpha+\beta}{2}-\alpha\right)$

$=P\sec\frac{\beta-\alpha}{2}=P\sec\frac{\alpha-\beta}{2}$

∴ co-ordinate of Point of intersection $=\left(P\sec\frac{\alpha-\beta}{2}, \frac{\alpha+\beta}{2}\right)$ (Proved)

13) Show that the straight line $r\cos\theta=\rho+a$ touches the circle $\cancel{\rho^2-2r\rho}\ r^2-2r\rho\cos\theta+\rho^2=a^2$. Find the point of contact.

⇒ The given eqⁿ's are,

$r\cos\theta=\rho+a$ ___ (i)

and $r^2-2r\rho\cos\theta+\rho^2=a^2$ ___ (ii)

from (i), $r=(\rho+a)\sec\theta$

putting this value into (ii),

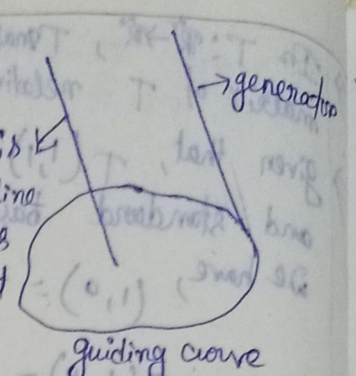
$$(\rho+a)^2\sec^2\theta-2(\rho+a)\rho\,\sec\theta\times\cos\theta+\rho^2=a^2$$
$$(\rho+a)^2\sec^2\theta-2(\rho+a)\rho+\rho^2-a^2=0$$
$$\text{or, }(\rho+a)^2\sec^2\theta-2(\rho+a)\rho+(\rho+a)(\rho-a)=0$$
$$\text{or, }(\rho+a)\sec^2\theta-2\rho+\rho-a=0$$
$$\text{or, }(\rho+a)\sec^2\theta-\rho-a=0$$
$$\text{or, }(\rho+a)\sec^2\theta-1=0$$
$$\text{or, }\sec^2\theta-1=0$$
$$\text{or, }\theta=0$$

∴ $r=\rho+\cancel{\rho}$

∴ The point of contact $=(\rho+a, 0)$

Cylinder

Definition: Cylinder is the surface generated by the movement of the straight line which is always parallel to a fixed straight line and intersect a given curve. The fixed straight line is called the axis of the cylinder, the curve is called the guiding curve and the moving straight line is called the generator.

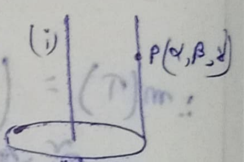


[Note: If the guiding curve be circle and the axis is perpendicular to the circle at centre then the cylinder is called right circular cylinder.]

1) Show that the eq^n of the cylinder whose generators are parallel to the straight line $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$ and whose guiding curve is the ellipse $x^2+2y^2=1,\ z=3$ is $3x^2+6y^2+3z^2+8yz-2zx+6x-24y-18z+24=0$.

$\Rightarrow$ The given eq^ns are,

$$\frac{x}{1}=\frac{y}{-2}=\frac{z}{3} \quad \text{---(i)} \quad \text{and} \quad x^2+2y^2=1,\ z=3 \quad \text{---(ii)}$$



Let, $P(\alpha,\beta,\gamma)$ be any point on the cylinder.

$\therefore$ The eq^n of the generator through this point is $\frac{x-\alpha}{1}=\frac{y-\beta}{-2}=\frac{z-\gamma}{3}$ ---(iii)

Putting $z=3$ into (iii), we have,

$$\frac{x-\alpha}{1}=\frac{y-\beta}{-2}=\frac{3-\gamma}{3}$$

$\therefore x-\alpha=\frac{3-\gamma}{3}$ and $y-\beta=\frac{2\gamma-6}{3}$

or, $x=\frac{3\alpha+3-\gamma}{3}$ $\quad\therefore y=\frac{3\beta+2\gamma-6}{3}$

Putting the value of x and y into the first eq^n of (ii) we have,

$$x^2+2y^2=1$$

$$\left(\frac{3\alpha+3-\gamma}{3}\right)^2+2\left(\frac{3\beta+2\gamma-6}{3}\right)^2=1$$

$\therefore$ The eq^n of the cylinder, which is the locus of the point P, is

$$\left(\frac{3x-z+3}{3}\right)^2+2\left(\frac{3y+2z-6}{3}\right)^2=1$$

$$9x^2+z^2+9-6xz-6z+18x+2(9y^2+4z^2+36+12yz-24z-36y)=9$$

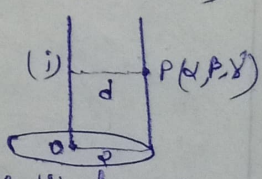
$$9x^2+z^2+\cancel{9}-6xz-6z+18x+18y^2+8z^2+24yz-48z=\cancel{9}$$
$$+72-72y$$

$$9x^2+18y^2+9z^2-6xz+24yz+18x-54z-72y+72=0$$

$$3x^2+6y^2+3z^2-2xz+8yz+6x-18z-24y+24=0 \quad \text{(Proved)}$$

2) Find the eq^n of the right circular ~~cone~~ cylinder whose axis is $\frac{x}{1}=\frac{y}{-2}=\frac{z}{2}$ and radius $=2$

The given eq^n is, $\frac{x}{1}=\frac{y}{-2}=\frac{z}{2}$ ---(i)



$\Rightarrow$ Let, $P(\alpha,\beta,\gamma)$ be any point on the cylinder.

Let, d be the perpendicular distance of the straight line (i) from P,

$\therefore d^2 = (\alpha^2+\beta^2+\gamma^2) - \left\{\frac{1\cdot\alpha - 2\beta + 2\gamma}{\sqrt{1^2+2^2+2^2}}\right\}^2 = (\alpha^2+\beta^2+\gamma^2) - \frac{(\alpha-2\beta+2\gamma)^2}{9}$

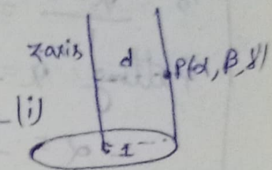
Now, by the given condition, $d = 2$

$\therefore d^2 = 4$

$\therefore 36 = 9(\alpha^2+\beta^2+\gamma^2) - (\alpha - 2\beta + 2\gamma)^2$

$\therefore$ The eqⁿ of the right circular cylinder, which is the locus of the point (α,β,γ), is $9(x^2+y^2+z^2) - (x-2y+2z)^2 = 36$

3) Find the eqⁿ of the right circular cylinder whose axis is z-axis and radius $= 1$



$\Rightarrow$ Let, $P(\alpha,\beta,\gamma)$ be any point on the cylinder.
Let, d be the perpendicular distance of the axis from the point P. We have the eqⁿ of z-axis, is, $\frac{x}{0} = \frac{y}{0} = \frac{z}{1}$ —(i)

$\therefore d^2 = (\alpha^2+\beta^2+\gamma^2) - (0\cdot\alpha + 0\cdot\beta + 1\cdot\gamma)^2$

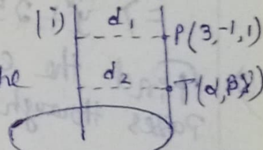
$= \alpha^2+\beta^2+\gamma^2-\gamma^2 = \alpha^2+\beta^2$

$\therefore$ By the given condition, $d = 1$, then $d^2 = 1$

$\therefore \alpha^2+\beta^2 = 1$

$\therefore$ The eqⁿ of the cylinder is, $x^2+y^2 = 1$. (Ans)

4) Find the eqⁿ of the ~~cyl~~ right circular cylinder which passes through the point $(3,-1,1)$ and has the straight line $\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z-2}{1}$ as it's axis.



$\Rightarrow$ The eqⁿ of the axis is, $\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z-2}{1}$ —(i)
Let, P be the point $(3,-1,1)$ and $T(\alpha,\beta,\gamma)$ be any point on the generator through P.
Let, d_1 and d_2 be the perpendicular distance of the straight line (i) from the points P and T respectively.

$\therefore d_1^2 = \{(3-1)^2+(-1+3)^2+(1-2)^2\} - \left\{\frac{2(3-1) - 1(-1+3) + 1(1-2)}{\sqrt{2^2+1+1^2}}\right\}^2$

$= (4+4+1) - \left\{\frac{4-2-1}{\sqrt{6}}\right\}^2 = 9 - \frac{1}{6} = \frac{53}{6}$

$\therefore d_2^2 = \{(\alpha-1)^2+(\beta+3)^2+(\gamma-2)^2\} - \left\{\frac{2(\alpha-1) - (\beta+3) + (\gamma-2)}{\sqrt{6}}\right\}^2$

$= (\alpha-1)^2+(\beta+3)^2+(\gamma-2)^2 - \frac{\{2(\alpha-1)-(\beta+3)+(\gamma-2)\}^2}{6}$

$= (\alpha-1)^2+(\beta+3)^2+(\gamma-2)^2 - \frac{(2\alpha-\beta+\gamma-7)^2}{6}$

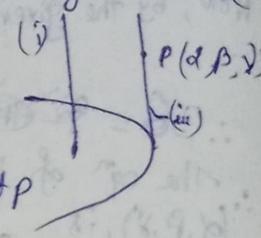
clearly, $d_1 = d_2$ $\therefore d_1^2 = d_2^2$

$\therefore (\alpha-1)^2+(\beta+3)^2+(\gamma-2)^2 - \frac{(2\alpha-\beta+\gamma-7)^2}{6} = \frac{53}{6}$

$\therefore$ The eqⁿ of the cylinder is, $(x-1)^2+(y+3)^2+(z-2)^2 - \frac{(2x-y+z-7)^2}{6} = \frac{53}{6}$

$6(x-1)^2+6(y+3)^2+6(z-2)^2 - (2x-y+z-7)^2 = 53.$

5) Find the eq^n of the cylinder generated by the straight line Parallel to the straight line $\frac{x}{1} = \frac{y}{5} = \frac{z}{-2}$ and which passes through the conic $x=0$, $y^2 = 6z$



$\Rightarrow$ Let, $P(\alpha, \beta, \gamma)$ be any point on the cylinder.

The given eq^{n}s are, $\frac{x}{1} = \frac{y}{5} = \frac{z}{-2}$ —(i)

and $x = 0$, $y^2 = 6z$ —(ii)

Now, the eq^n of the generator through the point P

is $\frac{x-\alpha}{1} = \frac{y-\beta}{5} = \frac{z-\gamma}{-2}$ —(iii)

Putting $x=0$ into (iii)

$$-\alpha = \frac{y-\beta}{5} = \frac{z-\gamma}{-2}$$

$\therefore y = \beta - 5\alpha$ and, $z = 2\alpha + \gamma$

Putting this values of y and z into the second eq^n of (ii)

$$y^2 = 6z$$

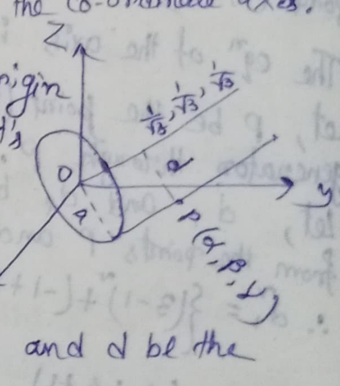
$$\Rightarrow (\beta - 5\alpha)^2 = 6(2\alpha + \gamma)$$

$\therefore$ The eq^n of the cylinder is, $(y - 5x)^2 = 6(2x + z)$

$$\Rightarrow y^2 + 25x^2 - 10xy = 12x + 6z$$

$$\Rightarrow y^2 + 25x^2 - 10xy - 12x - 6z = 0$$

6) Find the eq^n of the right circular cylinder of radius 'a' whose axis Passes through origin and makes equal angles with the co-ordinate axes.



$\Rightarrow$ Since, the axis of the cylinder Passes through origin and equally inclined to the Co-ordinate axis, it's eq^n is,

$$\frac{x-0}{1/\sqrt{3}} = \frac{y-0}{1/\sqrt{3}} = \frac{z-0}{1/\sqrt{3}}$$

$$\Rightarrow \frac{x}{1} = \frac{y}{1} = \frac{z}{1} \quad \text{—(i)}$$

Let, $P(\alpha, \beta, \gamma)$ be any point on the cylinder and d be the perpendicular distance of (i) from P.

$$\therefore d^2 = \{\alpha^2 + \beta^2 + \gamma^2\} - \left\{\frac{\alpha + \beta + \gamma}{\sqrt{3}}\right\}^2$$

$$= \alpha^2 + \beta^2 + \gamma^2 - \frac{(\alpha + \beta + \gamma)^2}{3}$$

By given condition, $d = a$ $\therefore d^2 = a^2$

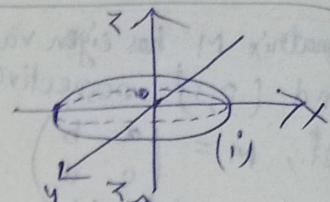
$$\therefore \alpha^2 + \beta^2 + \gamma^2 - \frac{(\alpha + \beta + \gamma)^2}{3} = a^2$$

$\therefore$ The eq^n of the cylinder, $3x^2 + 3y^2 + 3z^2 - (x + y + z)^2 = 3a^2$

$$\Rightarrow 2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2zx - 3a^2 = 0$$

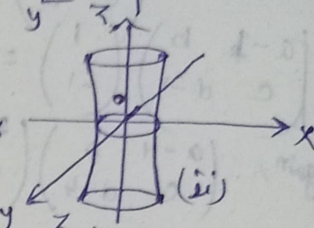
Generating line

i) **Ellipsoid :-** $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$



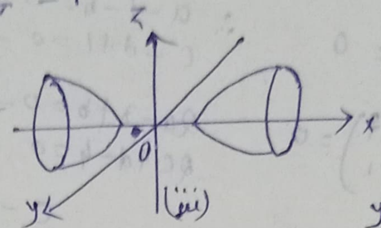
ii) **Hyperboloid of one sheet :-**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$



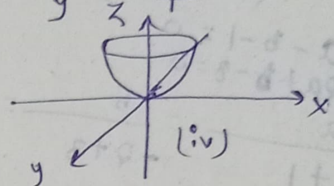
iii) **hyperboloid of two sheet :-**

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$



iv) **Elliptic Paraboloid :-**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}$$



v) **hyperbolic paraboloid :-**

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c}$$

Generating line of hyperboloid of one sheet :-

Let, the eq^n of the hyperboloid of one sheet be $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ —(i)

from (i) we have,

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1-\frac{y^2}{b^2}$$

$$\left(\frac{x}{a}+\frac{z}{c}\right)\left(\frac{x}{a}-\frac{z}{c}\right)=\left(1+\frac{y}{b}\right)\left(1-\frac{y}{b}\right) \text{ —(ii)}$$

Let,
$$\left.\begin{aligned}\frac{x}{a}+\frac{z}{c}&=\lambda\left(1+\frac{y}{b}\right) \text{ —(iii)}\\ \frac{x}{a}-\frac{z}{c}&=\frac{1}{\lambda}\left(1-\frac{y}{b}\right) \text{ —(iv)}\end{aligned}\right\} \text{—(v)}$$

The line (v) is called the generating line of the hyperboloid of one sheet. This generators are called λ-system of generators.

Again Let,
$$\left.\begin{aligned}\frac{x}{a}+\frac{z}{c}&=\mu\left(1-\frac{y}{b}\right) \text{ —(vi)}\\ \frac{x}{a}-\frac{z}{c}&=\frac{1}{\mu}\left(1+\frac{y}{b}\right) \text{ —(vii)}\end{aligned}\right\} \text{—(viii)}$$

The generators (viii) of (i) are called μ-system of generators.

The generators in λ and μ-system of hyperboloid of one sheet intersects each other.

1) Show that the generators in λ and μ-system of the hyperboloid of one sheet

=> The generators in λ and μ-system of $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ are respectively

$$\left.\begin{aligned}\frac{x}{a}+\frac{z}{c}&=\lambda\left(1+\frac{y}{b}\right) \text{ —(i)}\\ \frac{x}{a}-\frac{z}{c}&=\frac{1}{\lambda}\left(1-\frac{y}{b}\right) \text{ —(ii)}\end{aligned}\right\} \text{—(A)} \qquad \left.\begin{aligned}\frac{x}{a}+\frac{z}{c}&=\mu\left(1-\frac{y}{b}\right) \text{ —(iii)}\\ \frac{x}{a}-\frac{z}{c}&=\frac{1}{\mu}\left(1+\frac{y}{b}\right) \text{ —(iv)}\end{aligned}\right\} \text{—(B)}$$

from (i) and (iii),

$$\lambda\left(1+\frac{y}{b}\right)=\mu\left(1-\frac{y}{b}\right)$$

$$\frac{y}{b}(\lambda+\mu)=\mu-\lambda$$

$$y=\frac{b(\mu-\lambda)}{(\mu+\lambda)}$$

Putting the value of y into (i),

$$\frac{x}{a}+\frac{z}{c}=\frac{2\mu\lambda}{\mu+\lambda} \quad \text{---(v)}$$

and Putting the value of y into (iv),

$$\frac{x}{a}-\frac{z}{c}=\frac{2}{\mu+\lambda} \quad \text{---(vi)}$$

$\therefore$ from (v) and (vi)

$$x=\frac{a(\mu\lambda+1)}{\mu+\lambda}$$

$\therefore$ subtracting (vi) from (v),

$$\frac{2z}{c}=\frac{2\mu\lambda-2}{\mu+\lambda}$$

$$z=\frac{c(\mu\lambda-1)}{\mu+\lambda}$$

$\therefore$ The point of intersection is $\left(\frac{a(\mu\lambda+1)}{\mu+\lambda}, \frac{b(\mu-\lambda)}{\mu+\lambda}, \frac{c(\mu\lambda-1)}{\mu+\lambda}\right)$

2) generators of hyperbolic Paraboloid:-

Let, the eq^n of hyperbolic Paraboloid be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$ ---(i)

$\therefore$ The generators of (i) in λ-system,

$$\cancel{\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{a}+\frac{y}{b}\right)=2z}$$

$$\left.\begin{aligned}\frac{x}{a}+\frac{y}{b}&=\lambda z \quad \text{---(ii)}\\ \frac{x}{a}-\frac{y}{b}&=\frac{2}{\lambda} \quad \text{---(iii)}\end{aligned}\right\} \text{---(A)}$$

again the generators of (i) in μ-system,

$$\left.\begin{aligned}\frac{x}{a}-\frac{y}{b}&=\mu z \quad \text{---(iv)}\\ \frac{x}{a}+\frac{y}{b}&=\frac{2}{\mu} \quad \text{---(v)}\end{aligned}\right\} \text{---(B)}$$

3) Show that any two generators from different system of the hyperbolic paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$ intersects each other.

$\Rightarrow$ The generators of hyperbolic paraboloid in λ and μ-system are respectively,

$$\left.\begin{aligned}\frac{x}{a}+\frac{y}{b}&=\lambda z \text{ ---(i)}\\ \frac{x}{a}-\frac{y}{b}&=\frac{2}{\lambda} \text{ ---(ii)}\end{aligned}\right\}\text{---(A)} \quad \text{and} \quad \left.\begin{aligned}\frac{x}{a}-\frac{y}{b}&=\mu z \text{ ---(iii)}\\ \frac{x}{a}+\frac{y}{b}&=\frac{2}{\mu} \text{ ---(iv)}\end{aligned}\right\}\text{---(B)}$$

from (i) and (iv),

$$z=\frac{2}{\lambda\mu}$$

Putting the value of z into (i), $\frac{x}{a}+\frac{y}{b}=\frac{2}{\mu}$ ---(v)

from (ii) and (v), $x=\frac{a(\lambda+\mu)}{\lambda\mu}$ and $y=\frac{b(\lambda-\mu)}{\lambda\mu}$

$\therefore$ The point of intersection of two generators from different systems $\left(\frac{a(\lambda+\mu)}{\lambda\mu}, \frac{b(\lambda-\mu)}{\lambda\mu}, \frac{2}{\lambda\mu}\right)$

4) Find the eq^{n}s of the generating lines of the hyperboloid $\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{16}=1$ which passes through the point $(2,3,-4)$.

$\Rightarrow$ The given eq^n is, $\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{16}=1$ ---(i)

(i) can be written as,

$$\left(\frac{x}{2}+\frac{z}{4}\right)\left(\frac{x}{2}-\frac{z}{4}\right)=\left(1+\frac{y}{3}\right)\left(1-\frac{y}{3}\right) \quad \text{---(ii)}$$

$\therefore$ The generators in λ and μ systems are respectively,

$\frac{x}{2}+\frac{z}{4}=\lambda\left(1+\frac{y}{3}\right)$ — (iii)
$\frac{x}{2}-\frac{z}{4}=\frac{1}{\lambda}\left(1-\frac{y}{3}\right)$ — (iv) $\Big\}$ — (A) and $\frac{x}{2}+\frac{z}{4}=\mu\left(1-\frac{y}{3}\right)$ — (v)
$\frac{x}{2}-\frac{z}{4}=\frac{1}{\mu}\left(1+\frac{y}{3}\right)$ — (vi) $\Big\}$ — (B)

since, the generator (A) passes through the point (2, 3, −4)

$\therefore$ from (iii), $\lambda=0$

~~$\therefore$ the generator is, $\frac{x}{2}+\frac{z}{4}=0$~~

Putting the value of λ into (A) we have,

$$\frac{x}{2}+\frac{z}{4}=0$$

and $1-\frac{y}{3}=0$

again since, the generator (B) passes through the point (2, 3, −4)

$\therefore$ from (vi), $\mu=1$

$\therefore$ The generator is $\frac{x}{2}+\frac{z}{4}=\left(1-\frac{y}{3}\right)$ and $\frac{x}{2}-\frac{z}{4}=1+\frac{y}{3}$

5) Show that the straight line $x-1=y-2=z+1$ lies entirely on the surface $z^2-xy+2x+y+2z-1=0$.

$\Rightarrow$ The given eq^{n}s are, $x-1=y-2=z+1$ — (i) and $z^2-xy+2x+y+2z-1=0$ — (ii)

Any point on the straight line (i) is $(r+1, r+2, r-1)$

Now, $(z^2-xy+2x+y+2z-1)_{(r+1,\,r+2,\,r-1)}$

$=(r-1)^2-(r+1)(r+2)+2(r+1)+r+2+2(r-1)-1$

$=\cancel{r^2}+\cancel{1}-\cancel{2r}-\cancel{r^2}-\cancel{2r}-\cancel{r}-\cancel{2}+\cancel{2r}+\cancel{2}+\cancel{r}+\cancel{2}+\cancel{2r}-\cancel{2}-\cancel{1}$

$0=0$

$\therefore$ The straight line (i) entirely lies on the surface (ii).

6) Find the locus of the point of intersection of the perpendicular generators of the hyperbolic paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$.

$\Rightarrow$ The given eq^n is, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$ — (i)

The generators of (i) in λ and μ-system are respectively,

$\frac{x}{a}+\frac{y}{b}=\lambda z$ — (ii)
$\frac{x}{a}-\frac{y}{b}=\frac{2}{\lambda}$ — (iii) $\Big\}$ — (A) and $\frac{x}{a}-\frac{y}{b}=\mu z$ — (iv)
$\frac{x}{a}+\frac{y}{b}=\frac{2}{\mu}$ — (v) $\Big\}$ — (B)

The point of intersection of (A) and (B) are

$x=\frac{a(\lambda+\mu)}{\lambda\mu}$, $y=\frac{b(\lambda-\mu)}{\mu\lambda}$, $z=\frac{2}{\mu\lambda}$ — (vi)

Let, l_1, m_1, n_1 be the direction cosines of the generator (A).

$\therefore$ $\frac{l_1}{a}+\frac{m_1}{b}-\lambda n_1=0$ — (vii)

$\frac{l_1}{a}-\frac{m_1}{b}=0$ — (viii)

Solving (vii) and (viii) for l_1, m_1, n_1 by means of cross multiplication we get,

$$\frac{l_1}{\frac{\lambda}{b}}=\frac{m_1}{\frac{\lambda}{a}}=\frac{n_1}{\frac{2}{ab}}$$ — (ix)

Let, l_2, m_2, n_2 be the direction cosines of (B).

$\therefore \frac{l_2}{a} - \frac{m_2}{b} - \mu n_2 = 0$ —— (x)

$\frac{l_2}{a} + \frac{m_2}{b} + 0 \cdot n_2 = 0$ —— (xi)

Solving (x) and (xi) for l_2, m_2, n_2 by means of cross multiplication, we have,

$\frac{l_2}{\mu/b} = \frac{m_2}{-\mu/a} = \frac{n_2}{\frac{2}{ab}}$ —— (xii)

since, the generators in λ and μ-system are mutually perpendicular

$\therefore l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

$\Rightarrow \lambda\mu\left(\frac{1}{b^2} - \frac{1}{a^2}\right) + \frac{4}{a^2 b^2} = 0$

$\Rightarrow \lambda\mu(a^2 - b^2) + 4 = 0$

$\Rightarrow (a^2 - b^2)\frac{2}{z} + 4 = 0$ [By (vi)]

$\Rightarrow (a^2 - b^2) + 2z = 0$, which is the required locus.

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